Boundary-Value Problem for Two-Dimensional Fluctuations in Boundary Layers

Shunichi Tsugé
Consultant

and

Harold L. Rogler
United Research Corporation
428 Hill Street, Suite 21
Santa Monica, California 90405

July 1985

Final Report for Period March 1979 — May 1985

TECHNICAL REPORTS
FILE COPY

ARNOLD ENGINEERING DEVELOPMENT CENTER
ARNOLD AIR FORCE STATION, TENNESSEE
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE

This report has been reviewed by the Office of Public Affairs (PA) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.
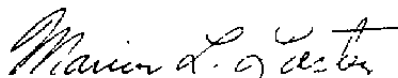
APPROVAL STATEMENT

This report has been reviewed and approved.

KEITH L. KUSHMAN
Directorate of Technology
Deputy for Operations

Approved for publication:

FOR THE COMMANDER

MARION L. LASTER
Director of Technology
Deputy for Operations

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION	1b RESTRICTIVE MARKINGS
UNCLASSIFIED	

2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT
	Approved for public release: distribution unlimited.
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	

4 PERFORMING ORGANIZATION REPORT NUMBER(S)	5 MONITORING ORGANIZATION REPORT NUMBER(S)
AEDC-TR-83-9	

6a NAME OF PERFORMING ORGANIZATION	6b OFFICE SYMBOL (If applicable)	7a NAME OF MONITORING ORGANIZATION
United Research Corporation		

6c ADDRESS (City, State and ZIP Code)	7b ADDRESS (City, State and ZIP Code)
428 Hill Street, Suite 21 Santa Monica, California 90405	

8a NAME OF FUNDING/SPONSORING ORGANIZATION	8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
Arnold Engineering Development Center	DO	F40600-79-C-0002

8c ADDRESS (City, State and ZIP Code)	10 SOURCE OF FUNDING NOS			
Air Force Systems Command Arnold Air Force Station, TN 37389-5000	PROGRAM ELEMENT NO	PROJECT NO	TASK NO	WORK UNIT NO
	65807F			

11 TITLE (Include Security Classification)
See Reverse of This Page.

12. PERSONAL AUTHOR(S)
Tsuge, Shunichi, Consultant and Rogler, Harold L., United Research Corporation

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Yr, Mo, Day)	15 PAGE COUNT
Final	FROM 3/79 TO 5/85	July 1985	52

16 SUPPLEMENTARY NOTATION
Available in Defense Technical Information Center (DTIC).

17	COSATI CODES		18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB GR	stability disturbances
20	.04		transition unsteady flow
14	02		eigenmodes standing waves (Cont)

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

The streamwise evolution of disturbances in a boundary layer is described as an asymptotic solution of the forced Orr-Sommerfeld equation. The velocity fluctuations and their derivatives are specified along the y-axis. With these boundary conditions, the effects are included of vortical and irrotational free stream disturbances, fluctuations originating from leading edges, and the discrete eigenmodes including the Tollmien wave. A Fourier transform in time and a Laplace transform in the streamwise direction are used. Complementary and particular integrals are found and the inverse transforms are taken. Five families of 2-D fluctuations can exist in a parallel-flow, incompressible boundary layer. Three families have exponentially growing fluctuations. One of these is the Tollmien stability wave. Another is an exponentially-growing standing wave that oscillates in time and does not travel. This fluctuation appears as a mathematical pole in transform space, like the stability waves, but does not vanish far away from the

(Cont)

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT	21 ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☒ DTIC USERS ☐	UNCLASSIFIED

22a NAME OF RESPONSIBLE INDIVIDUAL	22b TELEPHONE NUMBER (Include Area Code)	22c OFFICE SYMBOL
W. O. Cole	(615)454-7813	DOS

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE

11. Title

The Boundary-Value Problem for Two-Dimensional Fluctuations in Boundary Layers

18. Subject Terms (Concluded)

traveling waves   free stream disturbances
continuous spectrum   Orr-Sommerfeld equation
rotational disturbances   boundary-layer transition
irrotational disturbances   vorticity fluctuations

19. Abstract (Concluded)

boundary layer. A third growing fluctuation appears in Laplace space as a branch line. This continuous spectrum diffuses and travels upstream. The last two of these three growing fluctuations are excluded in our quarter-plane problem that extends forever downstream. Besides the Tollmien wave that can grow or decay in the streamwise direction, the other discrete modes appearing as mathematical poles are damped. Two other decaying fluctuations appear. One is a decaying standing wave which is the decaying counterpart to the growing standing wave. The other decaying fluctuation is the continuous spectrum with vortical fluctuations propagating downstream. In Laplace space, these last fluctuations appear as a branch line. All decaying fluctuations appear in our quarter-plane problem. This report serves as a roadmap for the five types of fluctuations. Other reports in this series describe the standing waves and the upstream-propagating continuous spectrum as solutions of the Orr-Sommerfeld equation. Still other reports show how free stream disturbances interacting with the leading edge can excite a spectrum of standing waves, and how surface waviness near a leading edge can induce a stationary version of the standing waves.

ACKNOWLEDGEMENTS

PREFACE

This report is one of several AEDC Technical Reports which describe the solutions of the Orr-Sommerfeld and Rayleigh equations and how these fluctuations are generated. These reports are:

Rotational and Irrotational Freestream Disturbances
Interacting Inviscidly with a Semi-Infinite Plate
AEDC-TR-83-3 April 1983

Exponentially-Varying, Unsteady Standing Waves
in Parallel-Flow Boundary Layers
AEDC-TR-83-4 May 1983

Nonperiodic Fluctuations Induced by Stationary Surface Waviness
on a Semi-Infinite Plate
AEDC-TR-83-10 July 1983

Waves Which Travel Upstream in Boundary Layers
AEDC-TR-83-7

Boundary-Value Problem
for Two-Dimensional Fluctuations in Boundary Layers
AEDC-TR-83-9 July 1985 (This report)

All reproducibles in this report were supplied by the author.

CONTENTS

APPENDIXES

ILLUSTRATIONS

TABLE

1.0 INTRODUCTION AND LITERATURE SURVEY

1.1 Introductory Comments

The streamwise evolution of 2-D unsteady fluctuations in a parallel-flow boundary layer is formulated in this analysis. The normal velocity fluctuation and derivatives are specified along the y-axis, and the behavior in the boundary layer downstream of the y-axis is obtained. Permissible disturbances include the Tollmien wave and other eigenwaves, but also include vortical and irrotational fluctuations in the freestream. The analysis links the boundary conditions with the amplitudes and phases of the various waves.

This boundary-value problem uses the asymptotic solution of the Orr-Sommerfeld equation by Tsuge (Ref. 1) which is summarized here in Appendix A. With this asymptotic theory, the Tollmien wave was calculated by Tsuge and Sakai (Ref. 2).

This report describes the details of Ref. 3 and is similar to the AIAA paper (Ref. 4).

For brevity, the list of initial-value problems in time (Section 1.2) and the list of boundary-value problems in space (Section 1.3) are restricted to problems that

    (a) analytically decompose initial or boundary conditions into the solutions of the Orr-Sommerfeld (or Rayleigh) equations, or those equations with a uniform mean flow

    (b) include freestream disturbances (i.e. attention is not restricted to the evolution of the discrete eigenmodes, except for channel flows where an infinite set of discrete eigenmodes is mathematically complete)

The Fourier-Laplace solutions of the Orr-Sommerfeld equation have been documented in other papers and reports. A summary of the known 2-D and 3-D solutions appears in Refs. 5b and 5c.

Besides the studies summarized in Sections 1.2 and 1.3, other approaches have been used to study unsteady shear layers, including

(1) Superpositions of eigenmodes with their amplitudes specified. References using this approach include Criminale (Ref. 6, Chpt. V), Gaster (Ref. 7), and Mack and Kendall (Ref. 8).

(2) Inviscid analyses of free-shear layers with step-function mean velocity profiles, e.g. Bechert (Refs. 9 and 10)

(3) Analytical solutions of the unsteady Prandtl boundary layer equations or higher-order boundary layer theory, as surveyed by Teleonis (Ref. 11)

(4) Lagrangian descriptions of turbulence convected past blunt bodies,

as described by the sudden distortion theories, e.g. Hunt (Ref. 12)

(5) Numerical solutions for shear layers using (a) the linearized, elliptic momentum equations with a specified mean flow, (b) the parabolized Navier-Stokes equations, (c) the Navier-Stokes equations, (d) Lagrangian tracking of vortex filaments, and (e) equations governing the evolution of the correlations with truncations or models for the higher correlations. Examples of these approaches are cited in Ref. 5c

(6) Compressible boundary-value or initial-value analyses with freestream disturbances by Tam (Refs. 13-15), Shapiro (Ref. 31), and Thomas and Lekoudis (Ref. 17)

(7) Deterministic experiments with vortical freestream disturbance by Dovgal, Kozlov, and Levchenko (Ref. 18) and Rockwell (Ref. 19).

1.2 List of Initial-Value Problems in Time for Parallel Flows

(1.2a) An array of vortices bisected by a plate for a general orientation of the vortices, including viscosity (Ref. 20; Ref. 21 is closely related mathematically)

(1.2b) 3-D oblique plane waves of vorticity bisected by a plate, including viscosity (Ref. 22)

(1.2c) Free-shear layer with uniform mean vorticity excited by vortical fluctuations inside and outside the shear layer; resonant and nonresonant excitation (Ref. 23). This study was documented in "The temporal response of a free-shear layer to vortical disturbances" (1978), an unpublished paper available from the author.

(1.2d) 3-D vorticity in a boundary layer represented by a layer of uniform mean vorticity; inviscid analysis by P. Durbin "Distortion of turbulence by a constant-shear layer adjacent to a wall," private communication (1977).

(1.2e) 2-D fluctuations in a boundary layer with two layers of uniform mean vorticity in unstable and stable configurations, inviscid analysis (Refs. 24 and 25)

(1.2f) Boundary layer with smooth velocity profiles, 2-D fluctuations, including viscosity (Ref. 26)

(1.2g) Nonlinear temporal evolution of longitudinal vortices in a parallel-flow boundary layer, inviscid (Ref. 20)

1.3 List of Boundary-Value Problems in Space for Parallel and Non-Parallel Flows

(1.3a) Downstream viscous 2-D half-plane problem with arrays of

vortices (Refs. 27 and 28)

(1.3b) Inviscid, uniform mean flow, 2-D fluctuations in a quarter-plane with a plate (Ref. 29)

(1.3c) Eight full-plane problems with a semi-infinite plate (with a leading edge), uniform mean flow (Refs. 30-33). These solutions are represented as a traveling wave and a superposition of standing waves in Ref. 34

(1.3d) Full-plane problem with a finite-length plate, inviscid, 2-D fluctuations (Ref. 30)

(1.3e) Inviscid problem with a mean boundary layer having two layers of constant mean vorticity (Ref. 35). This analysis shows the eigenmodes excited by vortical freestream disturbances.

(1.3f) Viscous, downstream quarter-plane problem with a smooth velocity profile, 2-D fluctuations (Refs. 3,4,36)

(1.3g) Viscous, downstream quarter-plane problem with a 2-D stationary wavy wall, 2-D fluctuations, smooth mean velocity profile (Ref. 37)

(1.3h) Flow past a semi-infinite, stationary wavy wall (Ref. 38)

(1.3i) Flow past a semi-infinite plate with traveling sinusoidally surface waves that travel at speeds different than the freestream speed, and galloping surface waves that travel at the freestream speed (Ref. 39)

(1.3j) Boundary-value problem for a channel (Ref. 40)

(1.3k) 3-D fluctuations in a viscous boundary-value problem with a smooth, parallel-flow boundary layer (Ref. 41)

(1.3l) Coupling between an oscillating freestream and a Tollmien-Schlichting wave in a nonparallel boundary layer (Ref. 42)

Other analyses have been carried out that link the initial conditions with amplitudes of the instabilities. These studies include nonresonant and resonant excitation of buoyancy instabilities (Refs. 43-45) and the temporal evolution of 3-D disturbances in an Ekman boundary layer (Ref. 46).

In the next section, the partial differential equation describing the evolution of disturbances will be derived and integral transformed to yield a forced Orr-Sommerfeld equation. In Section 3, an asymptotic solution of this forced equation is found. The inverse Laplace transform is obtained in Section 4. The formulation is summarized and discussed in Section 5. The method of successive approximation for solving the Orr-Sommerfeld equation is summarized in the appendix.

2.0 DERIVATION OF THE FORCED 3-D ORR-SOMMERFELD EQUATION

2.1 Introductory Comments

The three momentum equations for a constant property flow are (1) written with velocities and pressure separated into mean and fluctuation components and (2) averaged to obtain equations for the mean quantities. The averaged equations are subtracted from those of part (1) above to yield equations for the disturbances, and the disturbance equations are linearized for small-amplitude fluctuations. A parallel-flow, $\bar{U} = \bar{U}(y)$ and $\bar{V} = \bar{W} = 0$, is assumed. Derivatives of these momentum equations and the continuity equation are combined so that the pressure and two of the three velocities are eliminated. The resultant equation governing the evolution of 3-D fluctuations in a parallel flow is

$$\left\{ \frac{\partial}{\partial t} + \bar{U}(y)\frac{\partial}{\partial x} - \epsilon \nabla^2 \right\} \nabla^2 v - \bar{U}_{yy}(y)\frac{\partial v}{\partial x} = 0 \tag{2.1}$$

where $\epsilon = 1/R$ is the inverse of the Reynolds number.

This equation allows us to formulate a boundary-value problem in terms of solutions of the homogeneous Orr-Sommerfeld equation. This equation results when you seek solutions of form

$$v(x,y,z,t) = \varphi(y;\alpha,\omega,\gamma)e^{i\alpha x + i\gamma z - i\omega t} \tag{2.2}$$

where $\varphi(y)$ is a complex function, α and γ are the wavenumbers in the x and z-directions, and ω is the frequency. When (2.2) is introduced into (2.1), then the 3-D Orr-Sommerfeld equation results

$$\left\{ (\bar{U} - \frac{\omega}{\alpha})(D^2 - \alpha^2 - \gamma^2) - \bar{U}_{yy} - \frac{\epsilon}{i\alpha}(D^2 - \alpha^2 - \gamma^2)^2 \right\}\varphi = 0 \tag{2.3}$$

where $D = d/dy$ is the ordinary derivative.

Fourier-Laplace transforms of (2.1) yield not equation (2.3) but the forced version derived in Section 2.3. This procedure shows how to superimpose the solutions. Inversely, it shows how to decompose the initial disturbances into the waves.

Equations for primitive variables (such as velocities, vorticities, and pressure) are sometimes more useful than equations for the less intuitive variable φ. For example, equation (2.1) has a fourth derivative in x. Thus, four independent solutions are expected in the freestream as Ref. 27 indicated.

Also, the equation analogous to (2.1) for the 2-D vorticity $\zeta = v_x - u_y$

$$\left\{ \frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x} - \epsilon \nabla^2 \right\}\zeta + v\bar{\zeta}_y = 0 \tag{2.4}$$

indicates that the vorticity convects ($\zeta_x + \bar{U}\zeta_x$), diffuses ($\epsilon \nabla^2 \zeta$), and is produced ($v\bar{\zeta}_y$) in the shear layer. For inviscid flows with irrotational mean flows ($\bar{\zeta} = 0$) or flows with uniform mean vorticity ($\bar{\zeta}_y = 0$), the disturbance vorticity convects with the mean flow. For these flows, the vorticity is found by (analytically or numerically) tracking the vorticity as it convects downstream, followed by solving Poisson's equation for the disturbance streamfunction. By this method, solutions have been found which describe effects of leading and trailing edges, lateral edges, trailing vortex sheets, and other phenomena with practically important physics and mathematics. Although links are possible and desirable between these works and the Orr-Sommerfeld equation, direct study of the Orr-Sommerfeld equation has not provided the best insights into these new problems.

2.2 Domains for Boundary-Value Problems

Figure 2.1 shows the quarter-plane region ($x \geq 0$, $y \geq 0$) studied here. Sufficient conditions on the velocity fluctuation are specified along the y-axis, and along the plate coinciding with the x-axis, so that solutions can be obtained in that quadrant.

The domains for other boundary-value problems are also sketched in that figure.

2.3 Integral Transforms of the Partial Differential Equation

The coefficients $\bar{U}$ and $\bar{U}_{yy}$ in equation (2.1) are independent of x, z, and t. Thus, integral transforms in those directions reduce the partial differential equation to an ordinary differential equation. We focus on problems with "steady-state oscillations" where our wind tunnel has been turned on for a long time and the conditions along the y-axis oscillate in time (but not necessarily sinusoidally). The transient response arising from the start-up procedure has vanished from the region of interest.

For this case, equation (2.1) is Fourier transformed in time. Because the z-domain extends from $-\infty < z < \infty$, a Fourier transform in z is taken. In Chapter 3, we study a 2-D flow. The complex Fourier transform in time and the z-direction is

$$\hat{v}(x, y, \gamma, \omega) = \int\int_{-\infty}^{\infty} v(x, y, z, t) e^{-i\gamma z + i\omega t} \, dz \, dt \qquad (2.5)$$

The signs in the exponential have been chosen so the resultant transform is consistent with the classical Orr-Sommerfeld equation. An alternate transform is the generalized Fourier transform

$$\hat{v} = \lim_{a, b \to \infty} \int\int_{-\infty}^{\infty} v(x, y, z, t) e^{-i\gamma z - \frac{z^2}{a^2} + i\omega t - \frac{t^2}{b^2}} \, dz \, dt \qquad (2.6)$$

By either transform, equation (2.1) reduces to

9
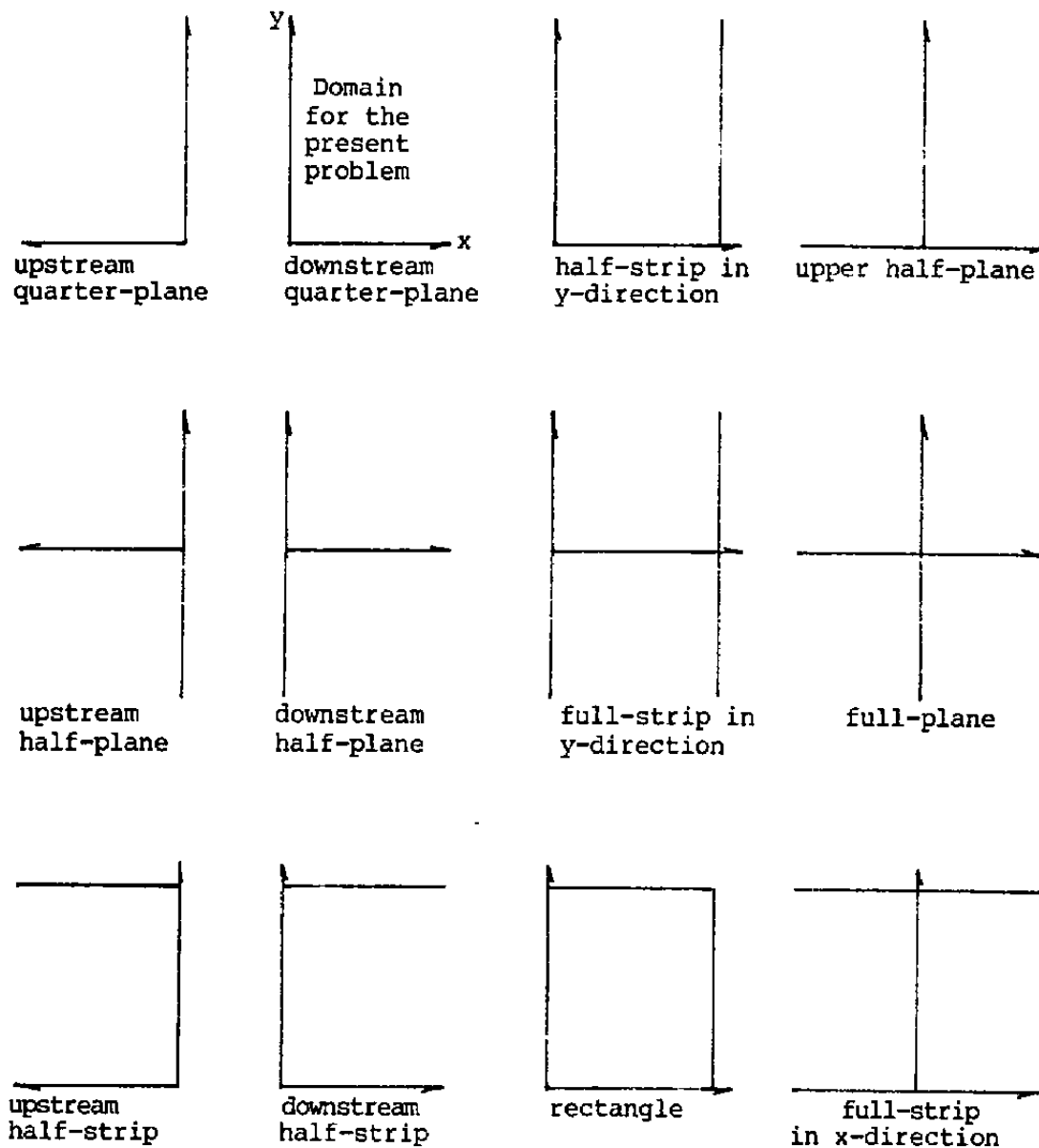
Figure 2.1   Some domains for 2-D boundary-value problems

$$\left\{-i\omega + \bar{U}\frac{\partial}{\partial x} - \epsilon\hat{\nabla}^2\right\}\hat{\nabla}^2\hat{v} - \bar{U}_{yy}\frac{\partial\hat{v}}{\partial x} = 0 \tag{2.7}$$

where the Laplacian reduces to

$$\hat{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \gamma^2 \tag{2.8}$$

Because we are interested in the downstream quadrant in Figure 2.1, a one-sided Laplace transform in the x-direction is taken

$$\varphi(y; s, \gamma, \omega) \equiv \int_0^\infty \hat{v}(x, y; \gamma, \omega) e^{-sx} dx \tag{2.9}$$

where s is the complex Laplace parameter. With this transformation, the following equation results

$$\left\{\left(\bar{U} - \frac{i\omega}{s}\right)(\rho^2 + s^2 - \gamma^2) - \bar{U}_{yy} - \frac{\epsilon}{s}(\rho^2 + s^2 - \gamma^2)^2\right\}\varphi = \frac{\mathcal{F}(y)}{s} \tag{2.10}$$

When $s = i\alpha$ is introduced, where α is the complex x-wavenumber, then the bracketed term on the left-hand side is the classical Orr-Sommerfeld operator in equation (2.3). The right-hand side is the forcing function

$$\frac{\mathcal{F}(y)}{s} = \left\{\left(\bar{U}\frac{\partial}{\partial x} - i\omega\right)\left(\frac{\partial}{\partial x} + s\right) + \bar{U}\left(\frac{\partial^2}{\partial y^2} + s^2 - \gamma^2\right) - \bar{U}_{yy}\right.$$
$$\left. - \epsilon\left(\frac{\partial^2}{\partial y^2} + s^2 - \gamma^2 + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} - \gamma^2\right)\left(\frac{\partial}{\partial x} + s\right)\right\}\frac{\hat{v}^{(0)}}{s} \tag{2.11}$$

where $\hat{v}^{(0)}$ is the normal velocity (or its derivatives) along the y-axis.

For a 2-D flow with $\gamma = 0$, the forcing function is

$$\frac{\mathcal{F}(y)}{s} = \left\{\left(\bar{U}\frac{\partial}{\partial x} - i\omega\right)\left(\frac{\partial}{\partial x} + s\right) + \bar{U}\left(\frac{\partial^2}{\partial y^2} + s^2\right) - \bar{U}_{yy} - \epsilon\left(\frac{\partial^2}{\partial y^2} + s^2 + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial}{\partial x} + s\right)\right\}\frac{\hat{v}^{(0)}}{s} \tag{2.12}$$

To introduce definiteness in our analyses, the forcing function is Fourier transformed and represented as

$$\mathcal{F}(y) = F^+(\beta) e^{i\beta y} + F^-(\beta) e^{-i\beta y} \tag{2.13}$$

Thus, the forced Orr-Sommerfeld equation for a 2-D flow studied in the next chapter is

$$\left\{\left(\bar{U} - \frac{i\omega}{s}\right)(\rho^2 + s^2) - \bar{U}_{yy} - \frac{\epsilon}{s}(\rho^2 + s^2)^2\right\}\varphi = \frac{\mathcal{F}^+ e^{i\beta y} + \mathcal{F}^- e^{-i\beta y}}{s} \tag{2.14}$$

where the Laplace parameter and the x-wavenumber are related by $s = i\alpha$.

11

3.0 SOLUTION OF THE TWO-DIMENSIONAL FORCED ORR-SOMMERFELD EQUATION

3.1 Particular Integral and Its Asymptotic Form

The objective now is to obtain the particular solution of the forced Orr-Sommerfeld equation

$$\left\{ (\bar{U} - c)(D^2 - \alpha^2) - \bar{U}_{yy} - \frac{1}{i\alpha R}(D^2 - \alpha^2)^2 \right\} \varphi = \frac{\mathcal{F}}{i\alpha} \tag{3.1}$$

where the forcing function is prescribed at $x = 0$ as the Fourier component

$$\mathcal{F}(y, \beta) = F^+(\beta)e^{i\beta y} + F^-(\beta)e^{-i\beta y} \tag{3.2}$$

The particular solution is obtained by the method of undetermined coefficients. For the Orr-Sommerfeld equation, the solution was given by Gustavsson (Ref. 26) as

$$\varphi_p = \frac{F}{W}\left\{ \varphi_1 \int \Delta_1 F dy + \varphi_2 \int \Delta_2 F dy + \varphi_3 \int \Delta_3 F dy + \varphi_4 \int \Delta_4 F dy \right\} \tag{3.3}$$

where W is the Wronskian

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ \varphi_1' & \varphi_2' & \varphi_3' & \varphi_4' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' & \varphi_4'' \\ \varphi_1''' & \varphi_2''' & \varphi_3''' & \varphi_4''' \end{vmatrix} \tag{3.4}$$

and Δ_j for $j = 1,2,3,4$ is the cofactor of φ_j''' with the sign reversed. φ_j for $j = 1,2,3,4$ are four independent solutions of the Orr-Sommerfeld equation.

Because the Orr-Sommerfeld equation lacks the next-to-highest derivative (the third derivative), then the Wronskian is constant. This means that the Wronskian only needs to be evaluated at one point, $y = y_1$.

The wall conditions and the asymptotic solutions for large y are listed in Table 1. The four solutions are derived in Appendix A. These asymptotic solutions are based on a new scaling, and are for flows where $0 \leq y < \infty$ or $-\infty < y < \infty$.

Substituting the asymptotic expressions for the φs, then the Wronskian is

$$W = 2\mu\alpha R^2(\alpha - \omega)^2 \qquad \text{where} \quad \mu = \left[\alpha^2 + iR(\alpha - \omega)\right]^{1/2} \tag{3.5}$$

Table 1.
Characteristics of the four independent,
asymptotic solutions of the Orr-Sommerfeld equation

Solution	Value at y=0	Asymptotic form for $y \gg \hat{y}$
ϕ_1	0	$(1-c)^{-1} \sinh\alpha(y-\hat{y})$
ϕ_2	0	$(1-c)\cosh\alpha(y-\hat{y})$
ϕ_3	1	$\hat{\phi_3} \exp[-\mu(y-\hat{y})]$
ϕ_4	0	$\hat{\phi_3}^{-1}\exp[+\mu(y-\hat{y})]$

where $\mu = [\alpha^2 + iR(\alpha - \omega)]$ and $\hat{\phi_3} = \phi_3(\hat{y})$

with the real part of μ chosen as positive. Also, the wall value of the particular solution (equation 3.3) must satisfy the impermeability condition, $\phi_p(0) = 0$, by choosing the lower bound of the integral as follows

$$\phi_p = \frac{R}{W}\left(\phi_1\int_{\hat{y}}^{y}\Delta_1 F\,dy + \phi_2\int_{\hat{y}}^{y}\Delta_2 F\,dy + \phi_3\int_{0}^{\infty}\Delta_3 F\,dy + \phi_4\int_{\infty}^{y}\Delta_4 F\,dy\right) \tag{3.6}$$

The asymptotic particular solution is obtained with the solutions in Table 1. The result is

$$(\phi_p)_{asym.} = \frac{i}{2(\alpha-\omega)}\left\{\frac{1}{\alpha(\alpha+i\beta)}\left[F^+e^{i\beta\hat{y}}\left(e^{\beta(y-\hat{y})}-e^{-\alpha(y-\hat{y})}\right)+F^-e^{-i\beta\hat{y}}\left(e^{-i\beta(y-\hat{y})}-e^{\alpha(y-\hat{y})}\right)\right]\right.$$

$$+\frac{1}{\alpha(\alpha-i\beta)}\left[F^+e^{i\beta\hat{y}}\left(e^{\beta(y-\hat{y})}-e^{\alpha(y-\hat{y})}\right)+F^-e^{-i\beta\hat{y}}\left(e^{-i\beta(y-\hat{y})}-e^{-\alpha(y-\hat{y})}\right)\right]$$

$$\left. -\frac{1}{\mu^2+\beta^2}\left(F^+e^{i\beta y}+F^-e^{-i\beta y}\right)\right\} \tag{3.7}$$

This solution is used later when an outer boundary condition is satisfied. Note that equation (3.7) includes terms which grow exponentially like $\exp(\alpha y)$. These terms are cancelled later so that the complete solution is well behaved. Equation (3.7) does not include terms like $\exp(\mu y)$.

3.2 Complementary Solution and the Assembly of the General Solution

The general solution of the forced Orr-Sommerfeld equation (3.1) is

$$\varphi = \varphi_p + \sum_{j=1}^{4} C_j \, \varphi_j \tag{3.8}$$

subject to the boundary conditions

$$\varphi = 0 \quad \text{(impermeability)} \; \left.\right\} \quad \text{at } y = 0 \tag{3.9}$$

$$\varphi' = 0 \quad \text{(no-slip)} \tag{3.10}$$

and φ is bounded as $y \to \infty$. $\tag{3.11}$

Because four of the five solutions satisfy impermeability

$$\varphi_p(0) = \varphi_1(0) = \varphi_2(0) = \varphi_4(0) = 0$$

then for the general solution (3.8) to satisfy impermeability, then the coefficient of the fifth term which does not satisfy impermeability must vanish

$$C_3 = 0 \tag{3.12}$$

Also, because only φ_4 grows like $\exp(\mu y)$ as $y \to \infty$, then the boundedness condition (3.11) requires that coefficient to vanish also

$$C_4 = 0 \tag{3.13}$$

The remaining constants, C_1 and C_2, are determined by the no-slip (equation 3.10) and boundedness (equation 3.11) conditions

$$C_1 \varphi_1'(0) + C_2 \varphi_2'(0) + \varphi_p'(0) = 0 \tag{3.14}$$

and

$$\frac{C_1}{1-c} + C_2(1-c) + \frac{i}{\alpha(\alpha-\omega)} \left[\frac{F^+ e^{i\beta \hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta \hat{y}}}{\alpha + i\beta} \right] = 0 \tag{3.15}$$

Equation (3.15) reflects the idea that the coefficient of $\exp(\mu y)$ vanishes in the asymptotic region. Note that φ does not vanish as $y \to \infty$, but oscillates as $\exp(\pm i\beta y)$.

These two equations for the two constants are solved to give

$$C_1 = \frac{1}{\Delta} \left[-\frac{\Phi(0)}{2\mu\alpha R} - i\varphi_2'(0) \left(\frac{F^+ e^{i\beta \hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta \hat{y}}}{\alpha + i\beta} \right) \right]$$

$$C_2 = \frac{1}{\Delta} \left[\frac{\alpha \Phi(0)}{2\mu R(\alpha-\omega)^2} + i\varphi_1'(0) \left(\frac{F^+ e^{i\beta \hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta \hat{y}}}{\alpha + i\beta} \right) \right] \tag{3.16}$$

14

where
$$\Phi'(0) = \varphi_1'(0) \int_{\hat{y}}^{0} \Delta_1 F \, dy + \varphi_2'(0) \int_{\hat{y}}^{0} \Delta_2 F \, dy + \varphi_4'(0) \int_{\infty}^{0} \Delta_4 F \, dy \qquad (3.17)$$

$$\Delta = \begin{vmatrix} \varphi_1'(0) & \varphi_2'(0) \\ \alpha^2 & (\alpha - \omega)^2 \end{vmatrix} \qquad (3.18)$$

Note that $\Delta = 0$ is identical with the eigenvalue condition for the discrete modes, as seen from

$$\varphi' = C_1 \varphi_1' + C_2 \varphi_2' = 0 \qquad \text{at } y = 0$$

and

$$\varphi = C_1 \frac{\alpha}{\alpha - \omega} e^{\alpha y} + C_2 \frac{\alpha - \omega}{\alpha} e^{\alpha y} = 0 \qquad \text{as } y \to \infty \qquad (3.19)$$

The impermeability condition is automatically satisfied by φ_1 and φ_2.

A more precise look at the behavior of reveals that must be rescaled in the non-asymptotic region. The behavior in the non-asymptotic region and the asymptotic (outer) region are

$$\varphi_1 = \alpha \left\{ y \left[B_0 + \alpha^2 B_1 + \alpha^4 B_4 + \cdots \right] + O(y^2) \right\} \qquad \text{for } y \ll \hat{y}$$

$$\varphi_1 \sim \frac{\sinh \alpha (y - \hat{y})}{1 - c} \qquad \text{for } y \gg \hat{y}$$

where B_0 is a constant depending only on $c = \omega/\alpha$; it is not affected by rescaling. Thus φ_1 must be rescaled so that

$$\varphi_1 \longrightarrow \alpha \varphi_1 \qquad (3.20)$$

for the non-asymptotic expression for φ_1.

The determinant Δ of equation (3.19) has multiple zeros corresponding to the eigenvalues of the discrete modes. The Tollmien-Schlichting mode is the lowest or fundamental mode. Then $1/\Delta$ is expanded as the partial fractions

$$\frac{1}{\Delta} = \sum_N \frac{h_N}{\alpha - \alpha_1^{(N)}} \qquad \text{with} \qquad h_N = \frac{1}{(\partial \Delta / \partial \alpha)_{\alpha = \alpha_1^{(N)}}} \qquad (3.21)$$

Using φ_1 as rescaled by equation (3.20), then

$$\frac{\partial \Delta}{\partial \alpha} = \frac{\partial}{\partial \alpha} \begin{vmatrix} \alpha \, \varphi_1'(0) & \varphi_2'(0) \\ \alpha^2 & (\alpha - \omega)^2 \end{vmatrix}$$

$$= \frac{\Delta}{\alpha} + \alpha \left\{ \begin{vmatrix} \dfrac{\partial \varphi_1'(0)}{\partial \alpha} & \dfrac{\partial \varphi_2'(0)}{\partial \alpha} \\ \alpha & (\alpha - \omega)^2 \end{vmatrix} + \begin{vmatrix} \varphi_1'(0) & \varphi_2'(0) \\ 1 & 2(\alpha - \omega) \end{vmatrix} \right\}$$

Because $\Delta = 0$ at $\alpha = \alpha_1^{(N)}$, then h_N is

$$h_N = \left[\alpha \begin{vmatrix} \partial \varphi_1'(0)/\partial \alpha & \partial \varphi_2'(0)/\partial \alpha \\ \alpha & (\alpha - \omega)^2 \end{vmatrix} + \alpha \begin{vmatrix} \varphi_1'(0) & \varphi_2'(0) \\ 1 & 2(\alpha - \omega) \end{vmatrix} \right]_{\alpha = \alpha_1^{(N)}}^{-1}$$

(3.22)

3.3 Summary of the solutions in (y,s,ω) Space

The final expression for the general solution of the Orr-Sommerfeld equation is

$$\varphi = \left(\sum_N \frac{h_N}{\alpha - \alpha_1^{(N)}} \right) \left\{ \left[-\frac{\alpha \hat{\Phi}(0)}{2\mu R} - i\alpha \varphi_2'(0) \left(\frac{F^+ e^{i\beta \hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta \hat{y}}}{\alpha + i\beta} \right) \right] \varphi_1 \right.$$

$$\left. + \left[\frac{\alpha^2 \hat{\Phi}(0)}{2\mu R (\alpha - \omega)^2} + i\alpha \varphi_1'(0) \left(\frac{F^+ e^{i\beta \hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta \hat{y}}}{\alpha + i\beta} \right) \right] \varphi_2 \right\}$$

$$+ \frac{\hat{\Phi}}{2\mu R (\alpha - \omega)^2}$$

(3.23)

The new φ_1 as rescaled in equation (3.20) is used. Because $\hat{\Phi}$ is linear in φ_1, it is rescaled by the same factor

$$(\varphi_1)_{old} \to \alpha \varphi_1 \qquad \text{and} \qquad \hat{\Phi}_{old} \to \alpha \hat{\Phi} \qquad (3.24)$$

Far from the wall $(y \gg y)$, an asymptotic general solution results when

the asymptotic expression (3.7) is substituted into equation (3.23). The result is

$$
\varphi = e^{-\alpha(y-\hat{y})}\left(\sum \frac{h_N}{\alpha - \alpha_i^{(N)}}\right)\left\{\left[\frac{\alpha}{2}\frac{1}{(\alpha-\omega)}\varphi_2'(0)\right.\right.
$$

$$
+ \frac{i}{2}(\alpha-\omega)\varphi_1'(0)\left]\left(\frac{F^+ e^{i\beta\hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta\hat{y}}}{\alpha + i\beta}\right) + \frac{\alpha}{(\alpha-\omega)}\frac{\hat{\Phi}'(0)}{2\mu R}\right\}
$$

$$
+ \frac{i}{2\alpha(\alpha-\omega)}\left\{\frac{1}{\alpha+i\beta}\left[F^+ e^{i\beta\hat{y}}\left(e^{i\beta(y-\hat{y})}- e^{-\alpha(y-\hat{y})}\right)+\frac{F^-}{2}e^{-i\beta y}\right]\right.
$$

$$
+ \frac{1}{\alpha-i\beta}\left[F^+ e^{i\beta y}+\frac{F^-}{2}e^{-i\beta\hat{y}}\left(e^{-i\beta(y-\hat{y})}- e^{-\alpha(y-\hat{y})}\right)\right]-\frac{\alpha F}{\mu^2+\beta^2}\right\}
$$

$$
(3.25)
$$

4.0 INVERSE LAPLACE TRANSFORM

4.1 General Expression for the Contributions from Poles and Branch Lines

In Chapter 3, the general solution of the forced Orr-Sommerfeld equation in transform space was obtained. To express the fluctuation velocity as a function of x and y, the inverse Laplace transform is necessary

$$\hat{V} = \frac{1}{2\pi i} \int_{\Gamma} \varphi e^{sx} ds$$

(4.1)

where Γ consists of the contours $s = s_0 + i\lambda$ and a semi-circle of infinite radius in the domain of Real(s) $\leq$ 0. The real value s is chosen so that the poles of φ are located on the left side of the line $s_0 + i\lambda$, with $-\infty < \lambda < \infty$. Figure 4.1 shows this contour, the poles, and the branch lines.

The contribution from the jth-pole, $s = s_j$, is calculated from the Cauchy theorem

$$I^{(j)} = \left[e^{sx} \varphi(s - s_j) \right]_{s = s_j}$$

(4.2)

If φ is not single-valued in the s-plane, as caused by fractional powers of s, then a Riemann cut is required. Because the integration contour Γ cannot pass through the Riemann cut, it must detour along the branch cuts, γ, or along another path which does not cross those cuts. This rule gives us the following expression

$$\hat{V} + \sum_j \frac{1}{2\pi i} \int_{\gamma_j} e^{sx} \varphi \, ds = \sum I^{(j)}$$

(4.3)

The contributions from the poles and branch line is obtained in the following sections.

4.2 Interior Solution

4.2.1 Poles from the Discrete Eigenmodes

Eqn. (3.23) for the interior solution shows that there are N+2 poles corresponding to $s = i\alpha_i^{(H)}$ and $s = \pm\beta$ if the forcing function $F^{\pm}$ is a pole-free analytic function. The equation also shows, for μ to have a positive real part, that branch lines are required because of the fractional power in eqn. (3.5). Because $F^{\pm}$ is open to free choice, poles or branch lines can be generated or eliminated by it.

The poles $s = i\alpha_i^{(N)}$ of eqn. (3.23) are for the discrete eigenmodes. According to the Cauchy theorem (4.2), we have
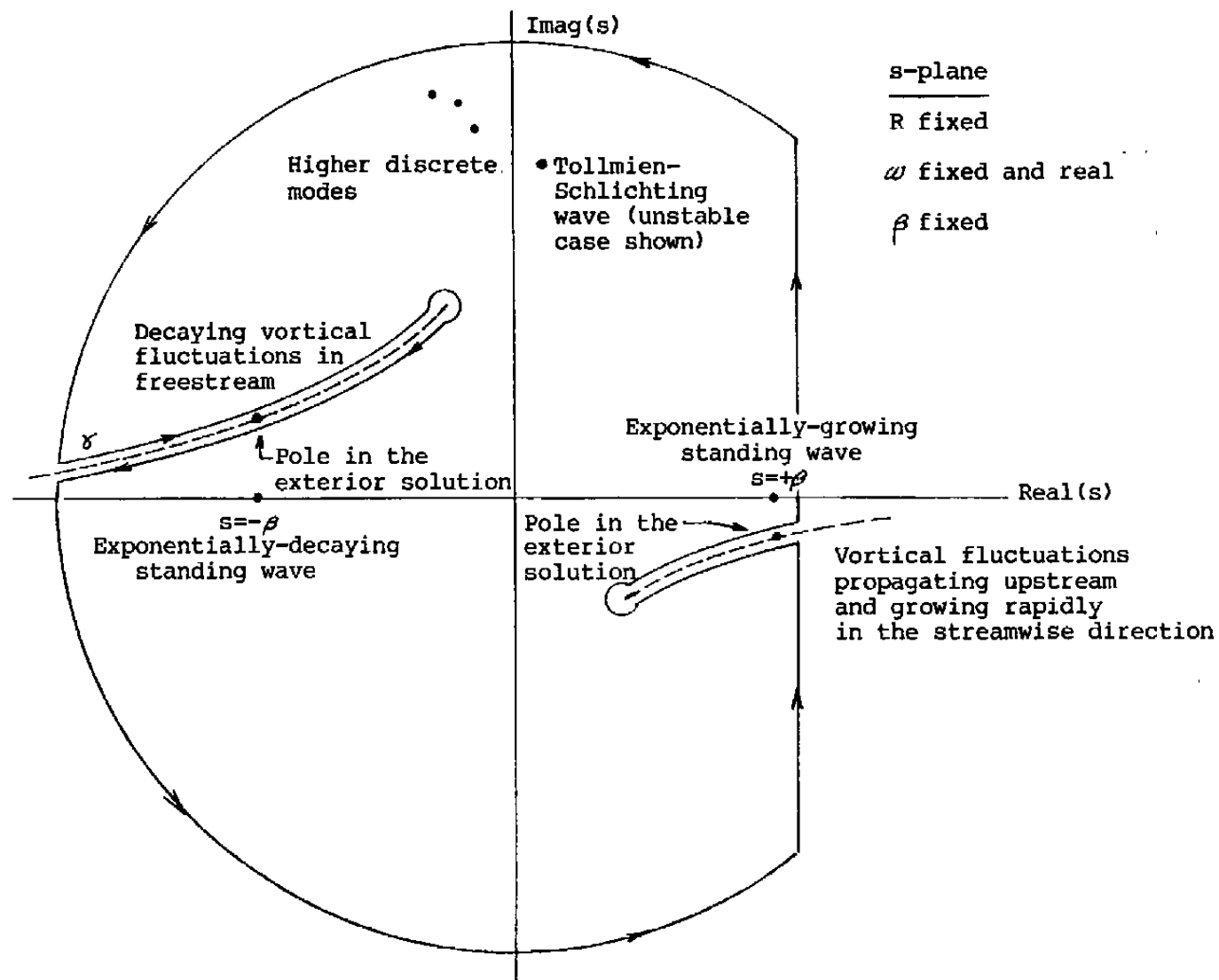
Figure 4.1 The contour taken in the inverse transform, showing the branch lines and poles.

$$
I_1^{(N)}(x,y) = e^{i\alpha_1^{(N)}x} \, i h_N \left\{ \left[-\frac{\alpha \, \bar{\Phi}(0)}{2\mu R} - i\alpha \phi_2'(0) \left(\frac{F^+ e^{i\beta \hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta \hat{y}}}{\alpha + i\beta} \right) \right] \phi_1(y) \right.
$$

$$
\left. + \left[\frac{\alpha^2 \, \bar{\Phi}'(0)}{2\mu R (\alpha - \omega)^2} + i\alpha \phi_1'(0) \left(\frac{F^+ e^{i\beta \hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta \hat{y}}}{\alpha + i\beta} \right) \right] \phi_2(y) \right\}_{\alpha = \alpha_1^{(N)}}
$$

<div align="right">(4.4)</div>

Each mode satisfies the impermeability and no-slip conditions independent of each other mode and independent of other contributions. These conditions are

$$
\left(I_1^{(N)} \right)_{y=0} = 0 \qquad \text{and} \qquad \left(\partial I_1^{(N)} / \partial y \right)_{y=0} = 0
$$

<div align="right">(4.5)</div>

The first condition is easily checked, $\phi_1(0) = \phi_2(0) = 0$. The second condition is fulfilled because of the eigenvalue condition

$$
\Delta = \begin{vmatrix} \alpha \, \phi_1'(0) & \phi_2'(0) \\ \alpha^2 & (\alpha - \omega)^2 \end{vmatrix}
$$

<div align="right">(4.6)</div>

which vanishes for $\alpha = \alpha_1^{(N)}$.

4.2.2 Poles from the Standing Wave Modes

The contribution from the pole $s = -\beta = s_z$ (or $\alpha = i\beta$) is also obtained in a straightforward manner. The result is

$$
I_2(x,y) = i \left(\sum \frac{h_N}{i\beta - \alpha_1^{(N)}} \right) e^{-\beta x} F^+ e^{i\beta \hat{y}} \left[-i\alpha \phi_2'(0) \phi_1 + i\alpha \phi_1'(0) \phi_2 \right]_{\alpha = i\beta}
$$

<div align="right">(4.7)</div>

This mode is a standing wave which decays as $\exp(-\beta x)$. It satisfies impermeability and no-slip. This mode was studied by Rogler and Reshotko in the freestream (Ref. 27) and appeared in another boundary-value problem in a quarter-plane (Ref. 29). Solutions of the Orr-Sommerfeld equation for boundary layers were calculated by Rogler and Tsuge (Ref. 5a) and Rogler (Refs. 5b-c) for this decaying wave and the corresponding growing wave. This growing wave is discussed next.

The contribution from the exponentially growing standing wave,

<div align="center">20</div>

$s = \beta$ (or $\alpha = -i\beta$) can be calculated in the same manner. However, this mode is suppressed in this quarter-plane problem where $x \rightarrow \infty$. To exclude this mode, the forcing function must have the form

$$F^- = (\alpha + i\beta)^m f^-$$

(4.8)

where m is an integer. Physically realizable situations exist where this growing mode is acceptable, such as the upstream influence of a trailing edge or another downstream boundary condition. The possibility of modeling such effects in a a problem with $0 < x < \infty$ is not considered here. This growing mode can appear in boundary-value problems where the x-domain is finite with $0 \leq x \leq L$, or semi-infinite with $-\infty < x \leq 0$, as Figure 2.1 shows.

4.2.3 Contributions from the Branch Line Yielding a Continuous Spectrum

As has been defined in eqn. (3.5), the function

$$\mu = (-s^2 + Rs - iR\omega)^{1/2}$$

(4.9)

is not single-valued in the s-plane. A branch cut is required which makes μ unique and maps the whole s-plane onto

$$Real(\mu) \geq 0$$

(4.10)

The contribution from this branch line is an integral over the continuous (vortical) mode. Condition (4.10) requires the branch cut to be a special curve. To simplify the task ahead, we let

$$\mu = [(s - s_3)(s_3' - s)]^{1/2}$$

(4.11)

where

$$s_3' = \frac{R + (R^2 - 4i\omega R)^{1/2}}{2} \approx R - i\omega$$

(4.12a)

$$s_3 = \frac{i\omega R}{s_3'} \approx i\omega - \frac{\omega^2}{R}$$

(4.12b)

and define

$$\sigma = s - s_3$$

(4.13)

Then

$$\mu = [\sigma(s_3' - s_3 - \sigma)]^{1/2}$$

(4.14)

Now consider a curve along which $Real(\mu) = 0$ (4.15)

Along that curve $arg\ \sigma + arg(s_3' - s_3 - \sigma) = \pm\pi$ (4.16)

The two vectors σ and $s_3' - s - \sigma$ and their arguments θ and θ' are shown

in Figure 4.2. The locus of the curve is obtained from the following relationship by taking the tangent of (4.16)

$$\frac{\sigma_i}{\sigma_r} + \frac{(s_3' - s_3)_i - \sigma_i}{(s_3' - s_3)_r - \sigma_r} = 0$$

or $\qquad \sigma_i = \dfrac{-\sigma_r(s_3' - s_3)_i}{(s_3' - s_3)_r - 2\sigma_r} \approx \dfrac{2\omega\sigma_r}{R - 2\sigma_r}$ $\qquad\qquad$ (4.17)

This branch line forms part of a hyperbola, labeled γ in Figure 4.1, along where μ is purely imaginary.

$$\mu = \frac{\pm i\tau^{1/2}[(s_r + 2\tau)^2 + s_i^2]^{1/2}(s_r + \tau)^{1/2}}{s_r + 2\tau}$$

$$\tau = \left| \sigma_r \right|$$

$$s_r = (s_3' - s_3)_r \approx R$$

$$s_i = (s_3' - s_3)_i \approx -2\omega \qquad\qquad (4.18)$$

Along its upper side, $\text{Imag}(\mu) > 0$ and along its lower side, $\text{Imag}(\mu) < 0$. The whole s-plane is mapped onto a half plane $\text{Real}(\mu) > 0$ by this branch cut. The contribution from the branch cut is

$$I_3(x,y) = \frac{1}{2\pi i} \int_\gamma e^{sx} \varphi \, ds$$

$$= \frac{1}{\pi i} \int_0^\infty \frac{d\tau(s_r + 2\tau) \exp\left\{[s_3 - \tau(1 - \frac{is_i}{s_r + \tau})]x\right\} g_3}{i\tau^{1/2}(s_r + \tau)^{1/2}[(s_r + 2\tau)^2 + s_i^2]^{1/2}}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.19)$

evaluated at $\alpha = -is_3 + i\tau\left(1 - \frac{is_i}{s_r + \tau}\right)$, and where g_3 is

$$g_3(y) = \left(\sum_N \frac{h_N}{\alpha - \alpha_i^{(N)}}\right)\left[-\frac{\alpha \, \Phi'(0)}{2R}\varphi_1 + \frac{\alpha^2 \, \Phi'(0)}{2R(\alpha - \omega)^2}\varphi_2\right] + \frac{\Phi}{2R(\alpha - \omega)^2}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.20)$
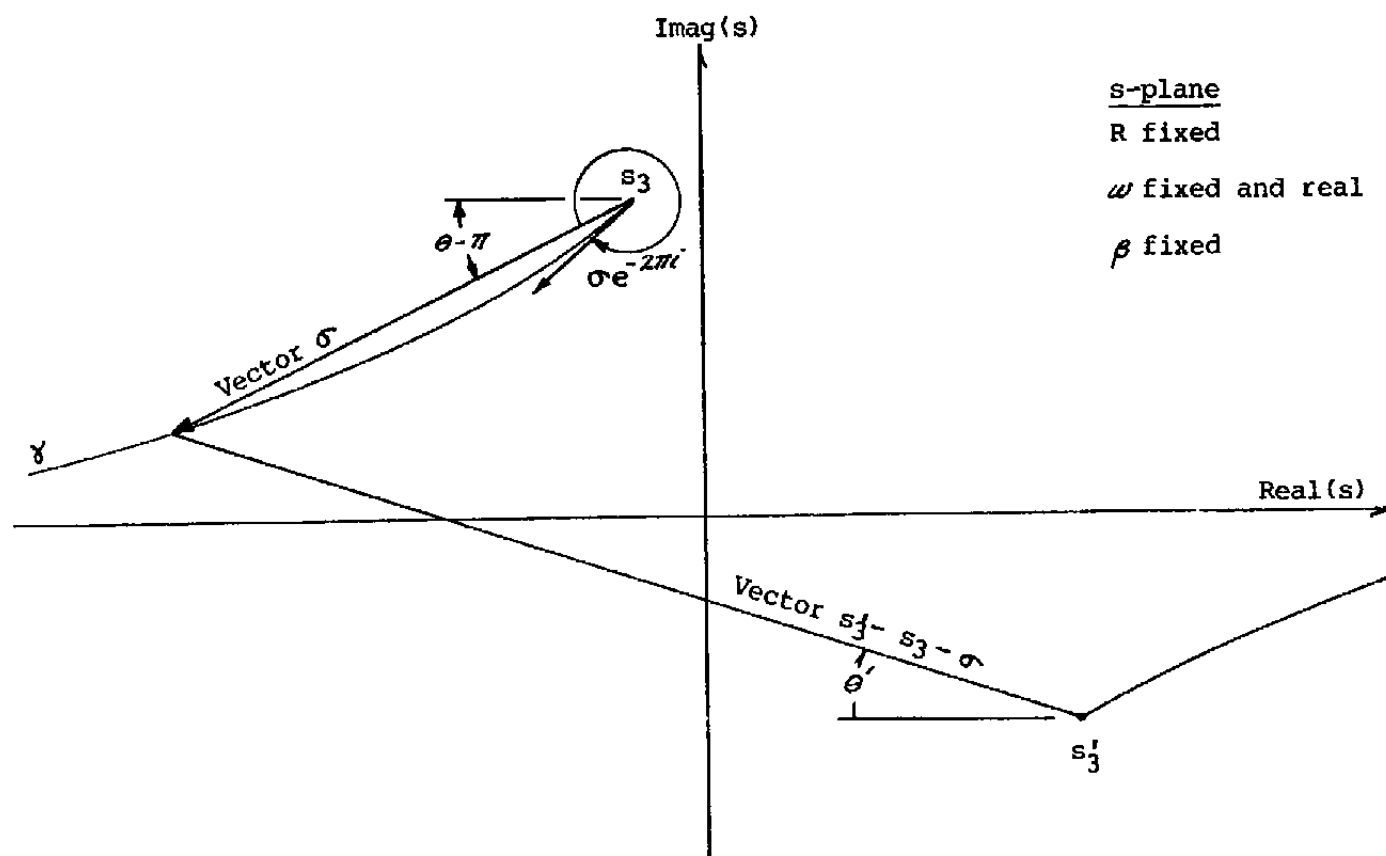
Figure 4.2 Geometry and notation for the branch lines associated with the
vortical fluctuations in the freestream.

Contribution, I_3, satisfies the impermeability and no-slip conditions

$$g_3(0) = 0 \qquad (\because \phi_1(0) = \phi_2(0) = \phi(0) = 0)$$

$$\left(\frac{dg_3}{dy}\right)_0 = \left(\sum_N \frac{h_N}{\alpha - \alpha_1^{(N)}}\right)\left(-\frac{\alpha\,\phi'(0)}{2R}\phi_1'(0) + \frac{\alpha^2\phi'(0)\phi_2'(0)}{2R(\alpha-\omega)^2}\right) + \frac{\phi'(0)}{2R(\alpha-\omega)^2}$$

$$= \frac{\phi'(0)}{2R(\alpha-\omega)^2}\left[1 + \left(\sum_N \frac{h_N}{\alpha - \alpha_1^{(N)}}\right)\left(-\alpha(\alpha-\omega)^2\phi_1'(0) + \alpha^2\phi_2'(0)\right)\right]$$

$$= \frac{\phi'(0)}{2R(\alpha-\omega)^2}\left[1 - \frac{\Delta}{\Delta}\right] = 0$$

For large Reynolds numbers, the terms $(s_i/s_r)^2 \sim O(R^{-2})$ are ignored. The expression simplifies to

$$I_3(x,y) = -\frac{e^{s_3 x}}{\pi}\int_0^\infty \frac{d\tau\,exp\left\{-x\tau\left(1 - \frac{is_i}{s_r+\tau}\right)\right\}g_3}{\tau^{1/2}(\tau+s_r)^{1/2}}\bigg|_{\alpha = -is_3 + i\tau\left(1 - \frac{is_i}{s_r+\tau}\right)}$$

(4.21)

The main part of the x-dependence of I_3 comes from the term $\exp(s_3 x)$. Thus, the contribution from the branch line is approximately

$$e^{-i\omega t}I_3 \sim exp\left[i\omega(x-t) - \frac{\omega^2 x}{R}\right]$$

(4.22)

where s_3 is given in eqn. (4.12). This approximate solution propagates with the freestream velocity and decays slowly in the x-direction. For $x \gg 1$, the x-dependence of the integral shows a universal feature. The contribution from the integrand is limited to a region close to $\tau \sim 0$. Transforming the variable from τ to t such that

$$t = \left[\tau x\left(1 - \frac{is_i}{s_r}\right)\right]^{1/2}$$

we have

$$I_3 \approx -\frac{e^{s_3 x}}{\pi\left[x\left(1 - \frac{is_i}{s_r}\right)\right]^{1/2}}\int_0^\infty \frac{dt\,e^{-t^2}}{s_r^{1/2}}(g_3)_{\alpha = -is_3}$$

$$= -g_3(\alpha = -is_3)e^{s_3 x}\bigg/\pi^{1/2}(s_r - is_i)^{1/2}x^{1/2}$$

(4.23)

which shows the enhancement of the spatial decay by the factor $x^{-1/2}$.

4.3 Exterior (Asymptotic) Solution

4.3.1 Poles from the Discrete Eigenmodes

The procedure used in Section 4.2.1 also applies to the asymptotic solution ϕ valid for $y \gg \hat{y}$ as given in eqn. (3.25)

$$I_,^{(N)}(x,y) = e^{-\alpha_,^{(N)}(y-\hat{y}) + i\alpha_,^{(N)}x} i h_N \left\{ \left[\frac{i\alpha}{2(\alpha-\omega)} \phi_2'(o) \right. \right.$$
$$\left. + \frac{i}{2}(\alpha-\omega)\phi'(o) \right] \left(\frac{F^+ e^{i\beta\hat{y}}}{\alpha - i\beta} + \frac{F^- e^{-i\beta\hat{y}}}{\alpha + i\beta} \right) + \frac{\alpha}{(\alpha-\omega)} \frac{\Phi'(o)}{2\mu R} \left. \right\}_{\alpha=\alpha_,^{(N)}}$$

$$(4.24)$$

This expression shows that this mode decays as $\exp(-\alpha_,^{(N)}y)$ toward the freestream as it should.

4.3.2 Poles from the Standing Waves

The asymptotic solution for the pole at $s = -\beta$ is

$$I_z(x,y) = i e^{-\beta x} \left\{ e^{i\beta(y-\hat{y})} \left(\sum \frac{h_N}{i\beta - \alpha_,^{(N)}} \right) F^+ e^{i\beta\hat{y}} \cdot \right.$$
$$\left. \frac{i}{2} \left[\frac{i\beta}{i\beta-\omega} \phi_2'(o) + (i\beta-\omega)\phi_,'(o) \right] + \frac{F^+ e^{i\beta\hat{y}}}{2\beta(i\beta-\omega)} \right. \qquad (4.25)$$

Because I_z has the form $\exp(-\beta x+i\beta y)$, far from the boundary layer edge, this mode is oscillatory in y and is irrotational. This agrees with Refs. 5a-c.

A similar expression can be obtained for the pole at $s = \beta$, corresponding to the $e^{\beta x}$ mode. If condition (4.8) applies, then this mode is suppressed in the asymptotic and general solutions.

4.3.3 Branch Line Yielding a Continuous Spectrum

The contribution from the branch line for the asymptotic solution is obtained from eqn. (4.19) with the function g_3 replaced by

$$(g_3)_{asym} = \left\{ e^{-\alpha(y-\hat{y})} \left(\sum_N \frac{h_N}{\alpha - \alpha_,^{(N)}} \right) \frac{\alpha \Phi'(o)}{(\alpha-\omega)2\mu R} \right\}$$

evaluated at $\quad \alpha = -i s_3 + i\tau \left(1 - \frac{i s_i}{s_r + \tau} \right) \qquad (4.26)$

Because $s_3 \approx i\omega - \omega^2/R$ from eqn. (4.12), this mode varies with y as

$$exp\left[-\alpha(y-\hat{y})\right] = exp\left[(y-\hat{y})\left(-\frac{i\omega}{R}^{2} - \omega + O(\tau)\right)\right]$$

(4.27)

where τ is the variable introduced in eqn. (4.18).

4.3.4 A Pole Representing a Solution Periodic in y and Slowly Decaying

The asymptotic expression (3.25) reveals a pole at

$$\mu^{2} + \beta^{2} = 0$$

(4.28)

which has no counterpart in the interior solution. It is located on the hyperbola branch line, and is the mode forced by the periodicity at the initial line, x = 0, and continues to y → ∞ with wavenumber β.

Eqn. (4.28) is rewritten as

$$\mu^{2} + \beta^{2} = -s^{2} + Rs - i\omega R + \beta^{2}$$
$$= -(s - s_{4})(s - s_{4}') = 0$$
$$s_{4} = i\omega - \frac{\omega^{2} + \beta^{2}}{R} + O(R^{-2})$$
$$s_{4}' = R - i\omega + O(R^{-1})$$

(4.29)

The contribution from the pole $s = s_{4}$ is

$$I_{4} = \frac{-F}{2(s_{4} - i\omega)(s_{4} - s_{4}')} e^{s_{4}x}$$

(4.30)

From eqn. (4.29), the asymptotic behavior is

$$e^{-i\omega\tau} I_{4} \sim exp\left[i\omega(x-\tau) \pm i\beta y - \frac{\omega^{2} + \beta^{2}}{R}x\right]$$

(4.31)

This solution is periodic in y, travels downstream with the freestream velocity, and slowly decays exponentially.

The same procedure for the pole at $s = s_{4}$ leads to a solution which grows like $exp(s_{4}'x)$. Although this mode can be physically realized as the upstream influence of a disturbance, it is excluded in this quarter-plane problem by factorizing the forcing function as

$$F = (s - s_{4}')^{n} G$$

(4.32)

where n is an integer.

4.3.5 The cases $\alpha = 0$ and $\alpha = \omega$

The exterior solution (3.25) has factors α and $\alpha - \omega$ in the denominator, and thus have pole-like characteristics. The first one, however, is ruled out because it gives a solution which approaches a nonvanishing constant

$$(\varphi)_{y \to \infty} = \frac{1}{2\omega\beta} \left[F^+ e^{i\beta\hat{y}} + F^- e^{-i\beta\hat{y}} \right] \neq 0$$

This reflects the fact that F should have a factor of α

$$F = \alpha G \tag{4.33}$$

in realistic situations, as inferred from the form of eqn. (3.1).

The other case of $\alpha = \omega$, or c = 1, should be treated separately. It is not covered by the present analysis. In that case, two solutions φ_3 and φ_4 need to be replaced by $y \exp(\pm\alpha y)$ which leads to a different expression for the Wronskian.

5.0 SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

5.1 Summary

The boundary-value problem in space for the evolution of two-dimensional fluctuations in a boundary layer is formulated. With velocity fluctuations specified along the y-axis, and impermeability and no-slip imposed along the plate (which lies along the x-axis), we seek the small-amplitude solutions in the quarter-plane downstream of the y-axis. These velocity fluctuations are functions of "exponential order" that can be Laplace transformed.

The differential equation

$$\left\{ \frac{\partial}{\partial t} + \bar{U}(y)\frac{\partial}{\partial x} - \epsilon \nabla^2 \right\} \nabla^2 v - \bar{U}_{yy}(y)\frac{\partial v}{\partial x} = 0$$

is Fourier transformed in time and Laplace transformed in the x-direction. The resultant forced Orr-Sommerfeld equation is

$$\left\{ (\bar{U} - \frac{\omega}{\alpha})(D^2 - \alpha^2) - \bar{U}_{yy} - \frac{1}{i\alpha R}(D^2 - \alpha^2)^2 \right\} \phi(y) = \frac{\mathcal{F}(y)}{i\alpha} \tag{5.1a}$$

where the forcing function is

$$\mathcal{F}(y) = \left(\bar{U} - \frac{i\omega}{s} - \frac{s}{R} - \frac{D^2}{sR} \right)\left(\frac{\partial}{\partial x} + s \right)\hat{v}^{(0)} + \left(\frac{i\omega}{s} + \frac{D^2}{sR} \right)\hat{\xi}^{(0)} + \frac{\hat{\xi}_x^{(0)}}{R} \tag{5.1b}$$

The forcing function is Fourier transformed in the y-direction and represented as

$$\mathcal{F} = \mathcal{F}^+(\beta)e^{i\beta y} + \mathcal{F}^-(\beta)e^{-i\beta y} \tag{5.2}$$

The particular integral of the forced equation is obtained by the method of variation of parameters.

The four independent solutions of the homogeneous equation are obtained as asymptotic solutions of the Orr-Sommerfeld equation with an asymptotic expansion in α/R where α is the x-wavenumber and R is the Reynolds number. The smallness of this parameter under ordinary conditions assures rapid convergence. These four independent solutions have unique properties exploited in the present solution.

The complementary and particular integrals are combined into a general solution which satisfies impermeability and no-slip at the wall and boundedness far-away.

The inverse Laplace transform is found via. the Cauchy theorem with a contour integration taken to the right of all poles, along a semi-circle of infinite radius, and with indentations about the branch lines.

The left and right handed branches of one Riemann cut are hyperbolas. Integration about the left handed cut yields the contribution from the slowly-decaying vortical fluctuations in the freestream.

Since contributions from the right-handed leg are physically unreasonable with a plate which extends to infinity downstream, conditions are imposed on the forcing function to eliminate those contributions.

Contributions from the isolated poles for the discrete modes are obtained from the Cauchy theorem. The amplitude of the Tollmien wave is found as the first in this series.

The amplitude of the decaying standing wave with another pole is similarly found.

The exponentially-growing standing wave is excluded by introducing a condition on the forcing function. This contribution is excluded for the same reason as given above for excluding contributions from the right-handed branch line.

The branch lines for the interior solution include the left and right-handed hyperbola branch cuts. The fluctuation associated with the right-handed branch line was calculated by Rogler (Ref. 47) as a solution of the Orr-Sommerfeld equation.

The poles in the interior solution include the N discrete modes, the two modes corresponding to the $\exp(\pm \beta x)$ standing wave modes, and poles along the hyperbola branch lines.

The poles in the asymptotic outer solution include the tails of the N discrete modes, the oscillatory solutions corresponding to $\mu^2 = -\beta^2$, and the oscillatory solutions corresponding to the standing waves with $s = \pm i\alpha$.

5.2 Procedure for Calculating the Initial Amplitude of the Tollmien Wave

(1) The procedure for calculating the four independent solutions of the homogeneous Orr-Sommerfeld equation, ϕ_1, ϕ_2, ϕ_3, and ϕ_4, is outlined in Appendix A and Ref. 1. These ϕs are solutions of the Orr-Sommerfeld equation, as asymptotic solutions or as numerical solutions with edge conditions defined by the solutions of Table 1. The derivatives $D\phi_1$ and $D\phi_2$ are evaluated at y=0, and the ϕs up to the 2nd derivative are stored for $0 \leq y < \hat{y}$.

(2) The derivatives $\partial(d\phi_1/dy)/\partial\alpha$ and $\partial(d\phi_2/dy)/\partial\alpha$ are evaluated at y = 0 and for $\alpha = \alpha_1^{(1)}$ corresponding to the Tollmien wave and stored.

(3) h_N is evaluated from equation (3.23).

(4) $\overset{\prime}{\oint}(0)$ is calculated from equation (3.17).

(5) $I_{i}^{(N)}(0,y)$ is evaluated from equation (4.4) for the amplitude of the Tollmien wave at x=0.

5.3 Conclusions and Recommendations

A boundary-value problem is formulated which illustrates how initial fluctuations are represented as a superposition of its (spatial) Fourier-Laplace solutions. Figure 4.1 serves as a roadmap for the five families of fluctuations which can appear in a boundary layer. The solution is complete in the sense that a superposition of these fluctuations completely describes the flowfield for an incompressible, viscous, small-amplitude, 2-D, unsteady flow in a parallel-flow boundary layer over a flat plate.

Since this formulation in 1980, these five classes were calculated as solutions of the Orr-Sommerfeld equation. Their streamlines are plotted in Figures B.1-B.5 in Appendix B of this report. A description of each fluctuation appears beside each figure. These descriptions provide an overview of the five families that serve as basic building blocks for fluctuations in boundary layers. The five are:

1. The discrete eigenmodes, where the Tollmien (or Tollmien-Schlichting) wave is the fundamental mode of boundary layer instability. These fluctuations vanish far from the boundary layer.

2. The slowly-decaying vortical fluctuations, with solutions described and calculated by Rogler and Reshotko (Ref. 48) and Salwen and Grosch (Refs. 49,50). Far from the boundary layer, these fluctuations are rotational.

3. The explosively-growing vortical fluctuations representing upstream diffusion of vorticity (Refs. 27,47). Far from the boundary layer, these fluctuations are rotational.

4. The exponentially-decaying standing waves described in Refs. 5a,b,c. In the freestream, these waves are irrotational. With the help of Mr. Arnie Rosner, a 16 mm animated movie was produced which illustrates these waves. The movie shows that these waves travel in the direction perpendicular to the wall. Ref. 34 shows that a spectrum of these standing waves appear downstream of the leading edge when freestream disturbances interact with that leading edge. The waves can be generated by surface waviness which is steady (Ref. 38) or unsteady (Ref. 39) if a leading edge exists.

5. The exponentially-growing standing waves also described in Refs. 5a,b,c. Although these waves are physically realizable in

some problems, they are excluded in this problem with a flat plate that extends to infinity downstream. Far from the boundary layer, these growing standing waves are irrotational.

In future studies, the theory could be applied to identify what properties of the initial disturbances are responsible for exciting the Tollmien waves, and the role of the mean velocity profile (as influenced by the mean pressure gradient and wall roughness). Further study is also required to account for the leading and trailing edges, 3-D disturbances, and the growth of the boundary layer.

The 3-D counterpart of this analysis was described in Ref. 41, but additional analysis and calculations are required to complete that study.

REFERENCES

1. Tsugé, S., "Methods of separation of variables in turbulence theory," NASA CR-3054 (1978).

2. Tsugé, S. and Sakai, H., "Stability characteristics of transitional and fully-developed flows with measured velocity profiles," Preprints to the Annual Meeting of the Japan Society for Aeronautical and Space Sciences, Paper I: page 30 (1981), Paper II: page 41 (1982).

3. Tsugé, S. and Rogler, H., "An asymptotic solution of the Orr-Sommerfeld equation based on a special coordinate stretching," Bulletin of the American Physical Society, Volume 25, Number 9 (1980).

4. Tsugé, Shunichi and Rogler, Harold L., "The two-dimensional, viscous boundary-value problem for fluctuations in boundary layers," AIAA-83-0044, AIAA 21st Aerospace Sciences Meeting, Reno, Nevada (10-13 January 1983).

5a. Rogler, H. and Tsugé, S., "Standing waves which decay or grow exponentially in the streamwise direction," Bulletin of the American Physical Society, Volume 26, Number 9 (1981).

5b. Rogler, Harold L., "Unsteady, exponentially-varying, standing waves in boundary layers," AIAA Paper 83-0045, AIAA 21st Aerospace Sciences Meeting, Reno, Nevada (10-13 January 1983).

5c. Rogler, Harold L., "Exponentially-varying, unsteady standing waves in parallel-flow boundary layers," AEDC-TR-83-4 (May 1983).

6. Betchov, Robert and Criminale, William O., Jr., Stability of Parallel Flows, Academic Press (1967).

7. Gaster, M., "Propagation of linear wave packets in laminar boundary layers," AIAA Journal, Volume 19, Number 4 (April 1981).

8. Mack, L.M. and Kendall, J.M., "Wave pattern produced by a localized harmonic source in a Blasius boundary layer," AIAA Paper 83-0046, AIAA 21st Aerospace Sciences Meeting, Reno, Nevada (10-13 January 1983).

9. Bechert, D. and Michel, U., "The control of a thin free shear layer with and without a semi-infinite plate by a pulsating flow field," Acustica, Volume 33, Number 5 (1975).

10. Bechert, D.W., "Excited waves in shear layers," DFVLR TR-82-23 (1982).

11. Telionis, Demetri P., Unsteady Viscous Flow, Springer-Verlag (1981).

12. Hunt, J.C.R., "A theory of turbulent flow round two-dimensional blunt bodies," Journal of Fluid Mechanics, Volume 61, Part 4, pages 625-706 (1973).

13. Tam, Christopher, "Excitation of instability waves in a two-dimensional shear layer by sound," Bulletin of the American Physical Society, Volume 22, Number 9 (November 1977).

14. Tam, Christopher, "Excitation of instability waves in a two dimensional shear layer by sound," Journal of Fluid Mechanics, Volume 89, pages 357-371 (1978).

15. Tam, Christopher K.W., "The excitation of Tollmien-Schlichting waves in low subsonic boundary layers by free-stream sound waves," Journal of Fluid Mechanics., Volume 109, pages 483-501 (1981).

16. Shapiro, P., "The influence of sound upon laminar boundary layer instability," Acoustics and Vibration Lab. Report No. 83458-83560-1, Massachusetts Institute of Technology (1977).

17. Thomas, A.S.W. and Lekoudis, S.G., "Sound and Tollmien-Schlichting waves in a Blasius Boundary layer," Physics of Fluids, Volume 21, pages 2112-2113 (1978).

18. Dovgal, A.V., Kozlov, V.V., and Levchenko, V.Ya., "Experimental investigation into the reaction of a boundary layer to external periodic disturbances," Fluid Mechanics (Russian), vol.15, no.4 (July-August 1980). Translated from Izvestiya Akademii Nauk SSSR, Mekhanika Zhidkosti i Gaza, Number 4, pages 155-159 (July-August 1980).

19. Rockwell, D., "Oscillations of impinging shear layers," AIAA Paper 82-0047, AIAA 20th Aerospace Sciences Meeting (11-13 January 1982).

20. Rogler, H., "The production of higher and lower modes and the decay of an array of vortices near a boundary," Bulletin of the American Physical Society, Volume 20, Number 9 (November 1975).

21. Hultgren, Lennart S. and Gustavsson, L. Hakan, "Algebraic growth of disturbances in a laminar boundary layer," Physics of Fluids, Volume 24, Number 6 (June 1981).

22. Rogler, Harold L., "Temporal decay of freestream vorticity disturbances near a wall," Consulting report for Dynatech, Inc. (1978).

23. Rogler, Harold L., "The Nonresonant and resonant responses of a free-shear layer to vortical and irrotational disturbances," Bulletin of the American Physical Society, Volume 22, Number 9 (November 1977).

24. Rogler, H.L., "The streamwise and lateral effects of freestream disturbances on boundary layers," *Bulletin of the American Physical Society*, Volume 21, Number 9 (1976).

25. Rogler, Harold L., "Time-varying response of Falkner-Skan boundary layers to rectangular vortices," *Proceedings of the 14th Annual Meeting of the Society of Engineering Science*, Lehigh University (14-16 November 1977).

26. Gustavsson, L. Håken, "Initial-value problem for boundary layer flows," *Physics of Fluids*, Volume 22, Number 9, pages 1602-1605 (1979).

27. Rogler, H. and Reshotko, E., "Spatially-decaying array of vortices," *Physics of Fluids*, Volume 19, Number 12 (December 1976).

28. Rogler, Harold L., "Spatially-decaying array of vortices," *Bulletin of the American Physical Society*, Volume 20, Number 9 (November 1975).

29. Rogler, Harold L., "Freestream vorticity disturbances adjusting to the presence of a plate -- a quarter-plane problem," *Journal of Applied Mechanics*, Volume 44, Number 4 (December 1977).

30. Rogler, Harold L., "The interaction between vortex-array representations of freestream turbulence and impermeable bodies," Paper 75-116, 13th AIAA Aerospace Sciences Meeting (January 1975).

31. Rogler, Harold L., "The interaction between vortex-array representation of freestream turbulence and semi-infinite flat plates," *Journal of Fluid Mechanics*, Volume 87, Part 3, pages 583-606 (1978).

32. Rogler, Harold L., "Rotational and irrotational freestream disturbances interacting with a semi-infinite plate," Eighth U.S. National Congress of Applied Mechanics, University of California at Los Angeles (26-30 June 1978).

33. Rogler, Harold L. and Reshotko, Eli, "Rotational and irrotational freestream disturbances interacting with a semi-infinite plate," AEDC-TR-83-3 (April 1983).

34. Rogler, H.L. and Reshotko, E., "The excitation of growing and decaying standing waves," *Bulletin of the American Physical Society*, Volume 27, Number 9 (1982).

35. Rogler, Harold L., "The coupling between freestream disturbances, driver oscillations, forced oscillations, and stability waves in a spatial analysis of a boundary layer," *Conference Proceedings of the Fluid Dynamics Panel Symposium on Laminar-Turbulent Transition*, Advisory Group for Aerospace Research and Development Conference Proceedings No. 224 (2-4 May 1977).

36. Grosch, Chester E., "Free-stream disturbances, continuous eigenfunctions, boundary-layer instability and transition," Report of the Old Dominion University Research Foundation, P.O. Box 6369 Norfolk, Virginia 23508 (September 1980).

37. Aldoss, Taha K. and Reshotko, Eli, "Contribution of roughness to disturbances in a boundary layer," Bulletin of the American Physical Society, Volume 27, Number 9 (1982).

38. Rogler, Harold L. and Chen, Chih-Tsai, "Nonperiodic fluctuations introduced by stationary surface waviness on a semi-infinite plate," AEDC-TR-83-10 (July 1983).

39. Park, Joel T., Silvus, H. Stanley, Cerwin, Steve A., and Rogler, Harold L. "Experiments on the influence of active walls on the initial amplitudes and growths of instability waves," Report SwRI 6984, Southwest Research Institute, 6220 Culebra Road, P.O. Drawer 28510, San Antonio, Texas 78284 (March 1985).

40. Schensted, I. V., "Contributions to the theory of hydrodynamic stability," Ph.D. Dissertation, U. of Michigan (1960).

41. Tsuge, S. and Rogler, H., "Initial-value problem and solutions for three-dimensional disturbances in boundary layers and free-shear layers," Bulletin of the American Physical Society, Volume 26, Number 9 (1981).

42. Goldstein, M.E., "Generation of instability waves at a leading edge," Paper 82-0926, AIAA/ASME 3rd Joint Thermophysics, Fluids, Plasma and Heat Transfer Conference, St. Louis, Mo. (7-11 June 1982).

43. Finucane, R.G. and Kelly, R.E., "Onset of instability in a fluid layer heated sinusoidally from below," International Journal of Heat and Mass Transfer, Volume 19, pages 71-85 (1976).

44. Kelly, R.E. and Pal, D., "Thermal convection with spatially periodic boundary conditions: resonant wavelength excitation," Journal of Fluid Mechanics (to be published).

45. Pal, R.E. and Kelly, R.E., "Thermal convection with spatially periodic nonuniform heating: nonresonant wavelength excitation," Proceedings of the 6th International Heat Transfer Conference, Toronto (1978).

46. Spooner, G. F. and Criminale, W.O, "The evolution of disturbances in an Ekman boundary layer," Journal of Fluid Mechanics, Volume 115, pages 327-346 (February 1982).

47. Rogler, Harold L., "Waves which travel upstream in boundary layers," AEDC-TR-83-7, July 1985.

48. Rogler, H.L. and Reshotko, Eli, "Disturbances in a boundary layer introduced by a low intensity array of vortices," _SIAM Journal of Applied Mathematics_, Volume 28, Number 2 (March 1975).

49. Grosch, C.E. and Salwen, H., "The continuous spectrum of the Orr-Sommerfeld equation. Part 1. The spectrum and the eigenfunctions," _Journal of Fluid Mechanics_, Volume 87, Part 1 (1978).

50. Salwen, Harold and Grosch, Chester E., "The continuous spectrum of the Orr-Sommerfeld equation. Part 2. Eigenfunction expansions," _Journal of Fluid Mechanics_, Volume 104, pages 445-465 (1981).

51. Lin, C.C., _The Theory of Hydrodynamic Stability_, Cambridge University Press (1955).

52. Heisenberg, W., "Über Stabilität und Turbulenz von Flüssigkeitsströmen" (On the Stability and Turbulence of Fluid Flows). _Ann. Phys._ Volume 74, pages 577-627 (1924).

Appendix A
METHOD OF SUCCESSIVE APPROXIMATION
FOR SOLVING THE ORR-SOMMERFELD EQUATION USING NEW SCALING

A.1 Introductory Comments

This appendix summarizes a method used to solve the Orr-Sommerfeld equation

$$\left\{ \frac{1}{i\alpha R} (D^2 - A^2)^2 - (\bar{U} - c)(D^2 - A^2) + \bar{U}_{yy} \right\} \varphi = 0 \tag{A.1}$$

where $D = d/dy$, $R = U_\infty \delta / \nu$ is the Reynolds number, α and γ are the dimensionless wavenumbers in the streamwise and spanwise directions, respectively, and $A^2 = \alpha^2 + \gamma^2$.

Heisenberg (Ref. 52) observed that the Orr-Sommerfeld equation can be solved analytically in the inviscid limit

$$\alpha R \rightarrow \infty \tag{A.2a}$$

with a small wavenumber $\quad A^2 \varphi \ll D^2 \varphi \tag{A.2b}$

Tsuge (Ref. 1) recognized that the reduced (but still viscous) Orr-Sommerfeld equation

$$\left\{ \frac{D^4}{i\alpha R} - (\bar{U} - c) D^2 + \bar{U}_{yy} \right\} \varphi = 0 \tag{A.3}$$

can be integrated once to yield a third-order equation

$$\left\{ \frac{D^3}{i\alpha R} - (\bar{U} - c) D + \bar{U}_y \right\} \varphi = C_1 \tag{A.4}$$

This equation is the zeroth-order equation associated with the following expansion

$$\varphi(y) = \varphi^{(0)}(\eta) + \frac{A^2}{\alpha R} \varphi^{(1)}(\eta) + \left(\frac{A^2}{\alpha R} \right)^2 \varphi^{(2)}(\eta) + \cdots \tag{A.5}$$

with $y = \epsilon \eta$ and $\epsilon = (\alpha R)^{-1/2} \tag{A.6a,b}$

The functions $\varphi^{(N)}$ (N=0,1,2,...) obey the following set of equations

$$L^{(0)} \varphi^{(0)} = 0 \tag{A.7}$$

$$L^{(0)} \varphi^{(1)} = L^{(1)} \varphi^{(0)} \tag{A.8}$$

$$L^{(0)} \varphi^{(n)} = L^{(1)} \varphi^{(n-1)} - \varphi^{(n-2)} \quad (n \geq 2) \tag{A.9}$$

where the operators $L^{(0)}$ and $L^{(1)}$ are defined by

$$L^{(0)} = d^4/d\eta^4 - i(\bar{U} - c) d^2/d\eta^2 + i\bar{U}_{yy} \tag{A.10}$$

$$L^{(1)} = 2\,d^2/d\rho^2 - i(\bar{U} - c)$$

(A.11)

Equation (A.4) is the zeroth-order equation in this set.

A.2 Four independent solutions of the zeroth-order equation

Equation (A.3 or A.8), or its integrated version (A.4)

$$\dddot{Y} - i(\bar{U} - c)\dot{Y} + i\dot{\bar{U}}Y = i\dot{C}$$

(A.12)

has four independent solutions, ϕ_j, $j=1-4$. Solutions ϕ_1 and ϕ_2 both tend to slowly varying functions, whereas ϕ_3 and ϕ_4 exhibit rapid decay or growth. The solutions are found in the order $\phi_3 \to \phi_2 \to \phi_4 \to \phi_1$, in a form where each solution requires at most knowledge of the ϕs from earlier solutions. ϕ_4 and ϕ_1 are found exactly in terms of ϕ_3 and ϕ_2.

Solution ϕ_3

The homogeneous solution of equation (A.4) is found first. Because of its prospective exponential decay, the solution ϕ_3 is assumed of form

$$\phi_3 = \exp \int_0^\rho \lambda \, d\rho$$

(A.13)

With this transformation of variables, the homogeneous form of equation (A.12) is

$$\ddot{\lambda} + 3\lambda\dot{\lambda} + \lambda^3 - i(\bar{U} - c)\lambda + i\dot{\bar{U}} = 0$$

(A.14)

or alternately and more conveniently

$$\dot{\lambda} = \mu - i(\bar{U} - c)$$
$$\dot{\mu} = -3\lambda\mu + 4i(\bar{U} - c)\lambda - \lambda^3$$

(A.15a,b)

Since we are seeking a solution of (A.15a,b) which decays exponentially in the asymptotic limit ($\dot{\lambda}, \dot{\mu} \sim 0$), the root for λ with negative real part is

$$\lambda \sim -e^{i\pi/4}(\bar{U} - c)^{1/2}$$
$$\mu \sim i(\bar{U} - c)$$
$$\left. \right\} \quad \text{for } \rho \gg 1$$

(A.16a,b)

In solving equation (A.15a,b) with asymptotic conditions (A.16a,b), it is useful to introduce a variable transformation

$$V = -\mu + 2i(\bar{U} - c) - \lambda^2$$
$$W = -2\mu + 3i(\bar{U} - c) - \lambda^2$$

(A.17a,b)

38

and work with the equations in the new dependent variables (V,W)

$$\dot{V} + \lambda V - 2i\,\dot{\bar{U}} = 0$$

$$\dot{W} + 2\lambda W - 3i\,\dot{\bar{U}} = 0 \qquad\qquad \text{(A.18a,b)}$$

subject to the asymptotic conditions

$$V, W \rightarrow 0 \quad as\ \eta \rightarrow \infty \qquad\qquad \text{(A.19a,b)}$$

The integration may be started at a point $\eta = \hat{\eta}$ ($\gg 1$) with the initial values

$$V = 2i\,\dot{\bar{U}}/\lambda$$

$$W = 3i\,\dot{\bar{U}}/2\lambda \qquad\qquad \text{(A.20a,b)}$$

and with λ given by equation (A.16a) are moderately small. The quadrature marches inward from this point, where the nonlinearity of equation (A.18) appears in the form

$$\lambda = -e^{i\pi/4}\left(\bar{U} - c + 2i\,V + i\,W\right)^{1/2} \qquad\qquad \text{(A.21)}$$

and is taken into account for $\eta < \hat{\eta}$.

This method of numerical integration of the original equation (A.15) works successfully only for the decaying solution φ_3. A formally identical procedure for the growing solution φ_4 suffers from numerical instabilities. The two solutions are different because the point $\eta = \infty$ is a saddle singularity of equation (A.15) for the case treated, but it is a nodal singularity from which infinitely many solutions emerge for φ_4.

A straightforward calculation from (A.14, A.21, and A.20) leads to the solution

$$\varphi_3(\eta) = \varphi_3(\hat{\eta})\left(\frac{\hat{\bar{U}} - c}{\bar{U} - c}\right)^{5/4} exp\left\{-e^{i\pi/4}\int_{\hat{\eta}}^{\eta}(\bar{U} - c)^{1/2}d\eta\right\}$$

$$\text{(for }\eta > \hat{\eta} \gg 1\text{)} \quad \text{(A.22)}$$

The classical counterpart of φ_3 given in Ref. 52 is

$$\varphi_3(\tau) = \int_{\infty}^{\tau}d\tau\int_{\infty}^{\tau}d\tau\,\tau^{1/2}H_{1/3}^{(1)}\left[\frac{2}{3}\left(i a_0\tau\right)^{3/2}\right] \qquad\qquad \text{(A.23)}$$

where $H_{1/3}^{(1)}$ is the Hankel function of the first kind and τ and a_0 are defined by

$$\tau = (y - y_c)(\alpha R)^{1/3}\ ,\quad a_0 = \left(\bar{U}_y\right)_c^{1/3} \qquad\qquad \text{(A.24)}$$

with subscript c signifying the value at the critical layer. For $\tau \gg 1$, (A.23) takes the form

$$\varphi_3(\tau) \sim \tau^{-5/4} \exp\left\{-e^{i\pi/4}\frac{2}{3}(a_0\tau)^{3/2}\right\}$$ (A.25)

If we employ the linear approximation (A.24) for $\bar{U}-c$ at the critical point, solution (A.22) reduces to (A.25).

Solution φ_2

After one of the solutions of the third order equation (A.13) is found, the difficulties in solving for the remaining solutions are considerably reduced because of the following theorem: If n independent solutions of the mth order linear differential equation are at hand, then the (n+1)th solution is obtained by solving an equation of (m-n)th order. To use this theorem for the second solution φ_2 requires that an equation of the order $3 - 1 = 2$ be derived by putting

$$\varphi_2 = \varphi_3 \int_0^\eta G \, d\eta$$ (A.27)

and by substituting into equation (A.13). This, in turn, is equivalent to claiming that if G is assumed in the form

$$G = \exp \int_0^\eta (-\lambda + S) \, d\eta$$ (A.28)

then the transformed variable S obeys a first order nonlinear equation. A simple calculation actually confirms the assertion, yielding a Riccati equation

$$\dot{S} + S^2 + \lambda S - W = 0$$ (A.29)

where λ and W have been solved from eqns. (A.8) and (A.11), respectively. Of the two asymptotic roots of (A.29) for $S \sim 0$, the one that vanishes as $\eta \to \infty$, namely

$$S = W/\lambda$$ (A.30)

is the correct choice, since only this root insures that φ_2 meets the need for having a slow variation as $\eta \to \infty$. In fact, then

$$\varphi_2 = \varphi_3 \int_0^\eta d\eta \exp\left\{\int_0^\eta (-\lambda + S) \, d\eta\right\} = \hat{\mathscr{E}}_2 (\bar{U}-c) \quad {}_{\eta > \hat{\eta} \gg 1}$$ (A.31)

with

$$\hat{\mathscr{E}}_2 = (\bar{U}-c)^{-3/2} \exp\left\{-\frac{i\pi}{4} + \int_0^{\hat{\eta}} S \, d\eta\right\}$$

40

Because the integrand in (A.31) is a rapidly growing function, only the portion near the upper bound contributes to the integral. Then the integral is replaced with

$$\left(-\lambda + s\right)^{-1} \exp\left\{ \int_0^\eta \left(-\lambda + s\right) d\eta \right\}$$

It is interesting to compare φ_2 of (A.31) with its classical counterpart

$$\left(\varphi_2\right)_{cl,in} = 1 + O(\alpha R)^{-1/3}$$

$$\left(\varphi_2\right)_{cl,out} = (\bar{U} - c) + O(\Delta^2) \tag{A.32}$$

which represent the inner (viscous) and outer (inviscid) solutions by Heisenberg (Ref. 52). The classical φ_2 is _uniformly_ slowly varying, whereas the φ_2 given here is moderately varying for $\eta \sim O(1)$, and is a rapidly varying function of the physical coordinate $y = \epsilon \eta$. This solution tends only asymptotically for $\eta \gg 1$ to a slowly varying function.

Also we note that solution (A.31) coincides with the leading term of the classical outer solution, which further substantiates the asymptotic method applied here.

Solution φ_4

Repeated use of the theorem assures that the third solution φ_4 satisfies a first order linear differential equation to be deduced from (A.13). This assertion is realized by the method of variation of constants

$$\varphi_4 = \Gamma_2 \varphi_2 + \Gamma_3 \varphi_3 \tag{A.33}$$

where Γ_2 and φ_3 are functions of η to be determined. Having eliminated φ_2 between the original equation (A.13) and the supplementary condition

$$\dot{\Gamma}_2 \varphi_2 + \dot{\Gamma}_3 \varphi_3 = 0$$

which is standard to this method, we are lead to the equation for Γ_3

$$\frac{\ddot{\Gamma}_3}{\dot{\Gamma}_3} = \frac{\ddot{\varphi}_2}{\dot{\varphi}_2} - 2 \frac{\varphi_2 \ddot{\varphi}_3 - \varphi_3 \ddot{\varphi}_2}{\varphi_2 \dot{\varphi}_3 - \varphi_2 \dot{\varphi}_3}$$

This equation is integrated twice, and its substitution into (A.33) gives

$$\varphi_4 = -\varphi_2 \int_0^\eta \varphi_3 \mathcal{W}^{-2} d\eta + \varphi_3 \int_0^\eta \varphi_2 \mathcal{W}^{-2} d\eta \tag{A.34}$$

where W is the Wronskian formed by ϕ_2 and ϕ_3

$$\mathcal{W} = \begin{vmatrix} \phi_2 & \phi_3 \\ \dot{\phi}_2 & \dot{\phi}_3 \end{vmatrix}$$

(A.35)

It is easily confirmed from solution (A.34) for ϕ_4 that it has an asymptotic form of exponential growth with η. In fact, using the same approximation as has been used previously in deriving (A.31), we have for (A.34)

$$\phi_4 \sim \frac{1}{2\lambda^8 \phi_2 \phi_3}$$

$$\sim \hat{\mathcal{B}}_4 (\bar{U}-c)^{-5/4} \exp \left\{ e^{\frac{i\pi}{4}} \int_{\hat{\eta}}^{\eta} (\bar{U}-c)^{1/2} d\eta \right\}$$

(A.36)

with

$$\hat{\mathcal{B}}_4 = \frac{e^{i\pi/4}}{2 \hat{\mathcal{B}}_2 \hat{\phi}_3 (\hat{\bar{U}}-c)^{5/4}}$$

In deriving the second row, the asymptotic expressions (A.24) and (A.31) for ϕ_2 and ϕ_3 have been used. Relationship (A.36) is again in qualitative agreement with its classical equivalent, as is the case for the decaying solution ϕ_2. The classical equivalent is, from Ref. 51,

$$\left\{ \phi_4 (\tau) \right\}_{cl} = \int_{-\infty}^{\tau} d\tau \int_{-\infty}^{\tau} d\tau \, \tau^{1/2} H_{1/3}^{(2)} \left\{ \frac{2}{3} (i a_0 \tau)^{3/2} \right\}$$

(A.37)

with the same nomenclature as before with the Hankel function of the second kind $H_{1/3}^{(2)}$. The asymptotic expression of (A.25) for $\tau \gg 1$ is

$$\left\{ \phi_4 (\tau) \right\}_{cl} \sim \tau^{-5/4} \exp \left\{ e^{\pi i/4} \frac{2}{3} (a_0 \tau)^{3/2} \right\}$$

(A.38)

in agreement with (A.36) with limited accuracy in the velocity profile.

The set of solutions (ϕ_2, ϕ_3, ϕ_4) which has been constructed has a noteworthy characteristic for the Wronskian that simplifies the analyses which follow

$$\mathcal{W}_{III} = \begin{vmatrix} \phi_2 & \phi_3 & \phi_4 \\ \dot{\phi}_2 & \dot{\phi}_3 & \dot{\phi}_4 \\ \ddot{\phi}_2 & \ddot{\phi}_3 & \ddot{\phi}_4 \end{vmatrix} = 1$$

(A.39)

This formula is easily checked by noting the following relationship

$$\frac{d \phi_4}{d\eta'} = -\frac{d \phi_2}{d\eta'} \int_0^\eta \frac{\phi_3}{W^2} d\eta + \frac{d \phi_3}{d\eta'} \int_0^\eta \frac{\phi_2}{W^2} d\eta + \frac{\delta_{l2}}{W}$$

which holds for $\mathcal{J} = 0, 1, 2.$

Solution φ_1

Since all homogeneous solutions are exhausted, the fourth solution φ_1 of equation (A.12) is the particular solution of the equation with $C_1 = 1$

$$\dddot{\varphi_1} - i(\bar{U}-c)\dot{\varphi_1} + i\dot{\bar{U}}\varphi_1 = i \tag{A.40}$$

The method of variation of constants is used here also, since all of the homogeneous solutions φ_2 through φ_4 are available. The theorem states that no differential equation needs to be solved. A manipulation using the key property (A.39) leads to the solution

$$\varphi_1 = i\varphi_2 \int_{\hat{\eta}}^{\eta} W I_3 \, d\eta - i\varphi_3 \int_0^{\eta} W I_2 \, d\eta + i\varphi_4 \int_\infty^{\eta} W \, d\eta \tag{A.41}$$

with $$I_\rho = \int_0^\eta \varphi_\rho W^{-2} \, d\eta \qquad (\mathcal{J} = 2,3) \tag{A.42}$$

where $\eta = \hat{\eta} \gg 1$ is a point beyond which the asymptotic expressions for φ_2 through φ_4 are valid. Taking the derivatives successively, we have

$$\frac{d^j \varphi_1}{d\eta^j} = i \frac{d^j \varphi_2}{d\eta^j} \int_{\hat{\eta}}^{\eta} W I_3 \, d\eta - i \frac{d^j \varphi_3}{d\eta^j} \int_0^{\eta} W I_2 \, d\eta$$

$$+ i \frac{d^j \varphi_4}{d\eta^j} \int_\infty^{\eta} W \, d\eta + i \delta_{3j} \quad (\mathcal{J} = 1,2,3) \tag{A.43}$$

This result can be checked by using (A.38), (A.32), and (A.33).

The function φ_1 in expression (A.41) behaves as a slowly varying function for $\eta \gg 1$ under the same conditions as invoked in deriving the asymptotic expression (A.31) for φ_2. The resulting expression is

$$\varphi_1 \sim -(\bar{U}-c) \int_{\hat{\eta}}^{\eta} (\bar{U}-c)^{-2} \, d\eta \qquad (\eta > \hat{\eta} \gg 1) \tag{A.44}$$

For comparison, the classical equivalent to this function is

$$(\varphi_1)_{cl,in} = \tau + O(\alpha R_\delta)^{-1/3}$$

$$(\varphi_1)_{cl,out} = (\bar{U}-c) \int_y (\bar{U}-c)^{-2} \, dy + O(A^2) \tag{A.45}$$

where τ is defined in (A.24). The leading terms of the asymptotic

expression agree with the leading terms of the classical __outer__ solution.

Because of (A.43), we can show the following relationship for the Wronskian formed by the four solutions φ_1 through φ_4

$$W_{IV} = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ \dot{\varphi}_1 & \dot{\varphi}_2 & \dot{\varphi}_3 & \dot{\varphi}_4 \\ \ddot{\varphi}_1 & \ddot{\varphi}_2 & \ddot{\varphi}_3 & \ddot{\varphi}_4 \\ \dddot{\varphi}_1 & \dddot{\varphi}_2 & \dddot{\varphi}_3 & \dddot{\varphi}_4 \end{vmatrix} \tag{A.46}$$

Higher Order Approximation

The full solution of the Orr-Sommerfeld equation is constructed from the series (A.6) by successive approximation starting with equation (A.13) with solution $\varphi^{(o)}$ known. In practice, the nth-order correction $\varphi^{(n)}$ is obtained by solving the inhomogeneous equations (A.10) following the same procedure as has lead from equation (A.40) to solution (A.41). The result is

$$\varphi_1^{(n)} = \varphi_2 \int_{\hat{\eta}}^{\eta} W I_3 J_1^{(n)} d\eta - \varphi_3 \int_0^{\eta} W I_2 J_1^{(n)} d\eta + \varphi_4 \int_\infty^{\eta} W J_1^{(n)} d\eta \tag{A.47}$$

with

$$J_1^{(n)} \equiv 2 \dot{\varphi}_1^{(n-1)} - i \int_{\hat{\eta}}^{\eta} (\bar{U}-c) \varphi_1^{(n-1)} d\eta - \int_{\hat{\eta}}^{\eta} \varphi_1^{(n-2)} d\eta \qquad \eta > \hat{\eta} \gg 1 \tag{A.48}$$

Using these formula, the same approximation as used in deriving the asymptotic expressions for φ_1 and φ_2, we get

$$\varphi_1^{(n)} \sim (\bar{U}-c) \int_{\hat{\eta}}^{\eta} \frac{d\eta}{(\bar{U}-c)^2} \int_{\hat{\eta}}^{\eta} (\bar{U}-c) \varphi_1^{(n-1)} d\eta \qquad (\text{for } \eta > \hat{\eta} \gg 1) \tag{A.49}$$

This expression, when substituted into A.5, with $\varphi^{(o)} = \varphi_j (j:1,2)$ given by (A.44) and (A.31) respectively, coincides with Heisenberg's inviscid outer solutions (Ref. 52). The series solutions for F_1 and F_2 are uniformly convergent if the lower bound $\hat{y}=\epsilon\hat{\eta}$ is taken such that $\bar{U}-c$ varies only slowly for $\hat{y} < y$. In fact, then, the series sum up to yield

$$F_1 = -(\bar{U}-c)^{-1} \sinh A(y-\hat{y}) \frac{(\alpha R)^{1/2}}{A} \tag{A.50}$$

$$F_2 = (\bar{U}-c) \cosh A(y-\hat{y}) \cdot \hat{e}_2 \tag{A.51}$$

In closing, this solution of the Orr-Sommerfeld equation for the Tollmien wave has been calculated and verified by Tsuge and Sakai (Ref. 2).

Appendix B
A summary of basic characteristics of five forms of 2-D spatial
oscillations in a parallel-flow, Blasius boundary layer, viscous, flat
plate, with $0 \leq y < \infty$

1. The Tollmien-Schlichting instability wave
is the fundamental wave in a finite set of
eigenmodes. The T-S wave is the only known
unstable eigenwave for an incompressible, flat
plate boundary layer. The higher modes are
all heavily damped for the Blasius layer. The
main physical features include two viscous
layers (a viscous sublayer near the wall, and
a critical layer near where $\bar{U}(y_c) =$Real(the
phase speed)). The T-S wave propagates
downstream and may grow or decay depending on
the boundary layer, Reynolds number, and
frequency. Across the sublayer, the Reynolds
stress jumps from zero to some value. Across
the critical layer, that stress jumps back to
zero. This Reynolds stress, if viscous
dampening is not too large, can lead to growth
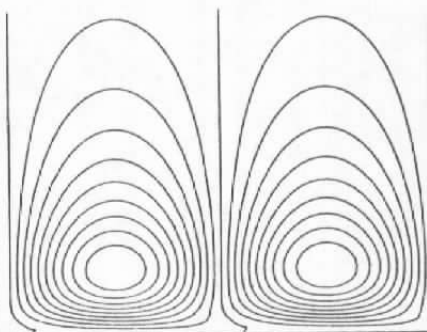of the wave.

Figure B.1

2. The decaying standing wave decays
exponentially in the streamwise direction and
oscillates sinusoidally in time as
$v = \varphi(y) \exp(-\beta x - i\omega t)$. This oscillation does not
propagate in the streamwise direction; the
phase speed is pure imaginary. Above the
boundary layer, the standing wave behaves as
$v = [A \exp(-my) + B \sin \beta y + \cos \beta y] \exp(-\beta x - i\omega t)$.
Far-away from the boundary layer, this
fluctuation is irrotational. The limit of a
standing wave as $\beta \to 0$ is a Stokes wave
$u(y,t) = f(y) \exp(-i\omega t)$, $v = 0$, for an oscillating
freestream. The "collision" of vortical and
other freestream disturbances with the leading
edge can excite a spectrum of these decaying
waves. Interactions between the trailing edge
and disturbances are also believed to excite
these waves. Steady waves of this form are
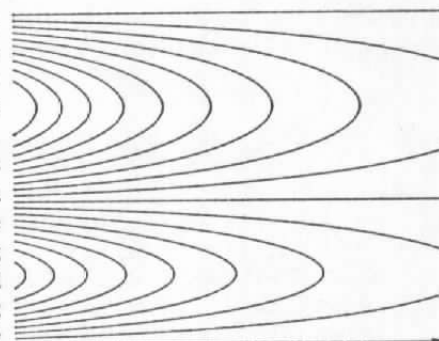also induced by flow over a wavy wall with a
leading edge.

Figure B.2

45

3. The <u>growing</u> <u>standing</u> <u>wave</u> is the exponentially growing counterpart to (2). If inviscid, oscillations (2) and (3) are related by $\varphi_3 = \varphi_2^*$. Although the viscous terms do not transform this way, calculations show that this relationship is a good guide for large Reynolds numbers. The steady limit of this wave behaves fundamentally different than (2). The phase speed is pure imaginary. This oscillation is believed to be excited by leading and trailing edges. Like (5), this oscillation represents an upstream influence of downstream b.c. in numerical calculations. Wall waviness can also excite these fluctuations.
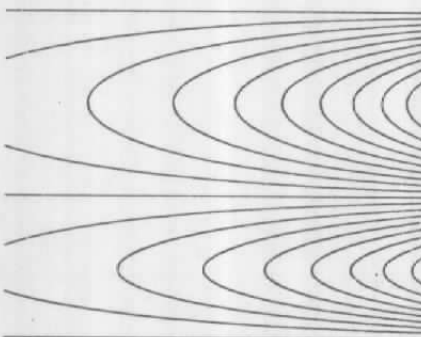

Figure B.3

4. <u>Decaying</u> <u>vortical</u> <u>fluctuations</u> propagate downstream with a speed slightly greater than the freestream, and decay slowly as $v = \varphi(y) \exp(i \alpha_r (x - c_{ep} t)) \exp(-\alpha_i x)$ where $\alpha_i = (\beta^2 + \omega^2)/R_\delta + \ldots$ A passive viscous sublayer forms at the wall. The smallness of the velocity fluctuations in the boundary layer is caused by the formation of a layer of vorticity at the "edge" of the boundary layer which induces a flow in the direction opposite to that induced by the vortical freestream disturbances. Outside of the boundary layer, the flow behaves as $v = [A \exp(-my) + B \sin \beta y + \cos \beta y] \exp[i \alpha (x - ct)]$. Far above the boundary layer, the fluctuations are <u>rotational</u>, in contrast to waves (2) and (3).
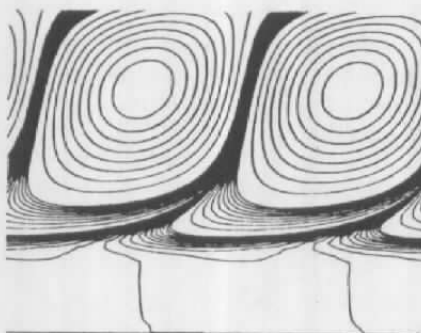

Figure B.4

5. The <u>upstream</u> <u>traveling</u> <u>vortical</u> <u>wave</u> grows explosively like $\exp(+R_\delta x)$. It propagates upstream at a speed approximately equal to $-U_\infty$ and represents upstream diffusion of vorticity. Difficult to calculate due to the very high frequency oscillation $\exp(i R_\delta y)$ in the boundary layer. This oscillation, like waves (2)-(4), also exists for a uniform mean flow, $\bar{U}=1$, in contrast to the stability waves for which a boundary layer must be present. Above the boundary layer, this oscillation behaves as $v = [A \exp(-my) + B \cos \beta y + \sin \beta y] \exp[i \alpha (x - ct)]$. This wave is one of the upstream influences of a downstream boundary condition in a calculational domain. It is one of the upstream influences of leading and trailing edges, and is perhaps an additional viscous diffusion effect in non-parallel flows.
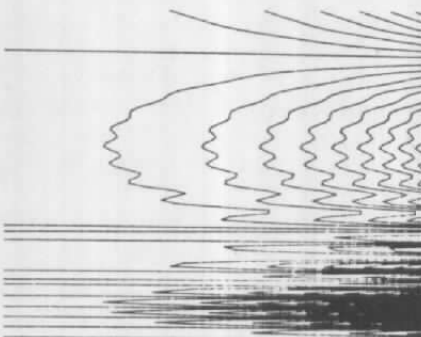

Figure B.5

NOMENCLATURE

English

a, b	convergence parameters in generalized Fourier transforms, equation (2.6)
a_o	defined in equation (A.24)
$A^2 = \alpha^2 + \gamma^2$	
B_i	constants in equation (3.19a)
$c = c_r + ic_i$	complex phase speed
C_j	complex constants in equation (3.8)
$D = d/dy$	ordinary derivative
$f(y) = f_r + if_i$	complex amplitude of the longitudinal velocity
F	forcing function
$F^-(\beta)$, $F^+(\beta)$	Fourier coefficients of the forcing function
g_a	defined in equation (4.20)
G	defined in equation (A.28)
h_N	constants in the partial fractions of equation (3.21)
i	$(-1)^{1/2}$
$I^{(j)}$	contribution from the jth pole as defined in equation (4.2)
$L^{(o)}$, $L^{(1)}$	operators defined in eqns. (A.11, A.12)
p	disturbance pressure
R, R_δ	Reynolds number based on characteristic thickness of the boundary layer
$s = i\alpha$	Laplace parameter
s_o	real value of s lying to the right of all poles
s_a, s_a'	values of s defined in equation (4.11)
S	transformed variable in equation (A.28)
t	time
u, v, w	disturbance velocities in the x, y and z directions
$\bar{U}(y)$	mean velocity in the streamwise direction
U_∞	mean x-velocity in the freestream
V, W	mean velocities in the y and z directions; see equation (A.17) for use of V, W as transformed variables in Appendix A
W	Wronskian
x	coordinate parallel to plate in the streamwise direction
y	coordinate normal to the plate
y_e	y-value of the boundary layer edge
Y	y-value for the asymptotic matching
z	spanwise coordinate

Greek and Script

α	x-wavenumber
$\alpha_i^{(N)}$	x-wavenumbers of the discrete modes
β	y-wavenumber
γ	z-wavenumber; contour of integration
∇^2	Laplacian operator
δ	characteristic thickness of the boundary layer; delta function
Δ_i	cofactor
Δ	defined in equation (3.18)
$\epsilon = 1/R$	inverse of Reynolds number

μ $[\alpha^2 + iR(\alpha - \omega)]^{1/2}$; see equation (A.15a,b)
        for definition in Appendix A

ν kinematic viscosity

ζ disturbance vorticity in the z-direction

$\phi(y) = \phi_r + i\phi_i$ complex amplitude of the normal velocity disturbance

$\phi_i, i=1,2,3,4$ four independent solutions of the Orr-Sommerfeld equation

$\phi'(0)$ defined in equation (3.17)

Γ integration contour

λ $-\infty < \lambda < \infty$, imaginary part of s lying to the right
        of all poles; variable defined in equation (A.13)

ω frequency

σ vector defined in equation (4.13)

τ defined in equation (4.18)

<u>Superscripts, Subscripts, and Miscellaneous Notation</u>

$(\bar{})$ time average

e   boundary layer edge

p   particular integral

r, i   real and imaginary parts of a complex variable

$(\;)_x$ partial derivative with respect to x

$\hat{}^x$ Fourier transform

(0)   evaluated at x=0

c.l.   classical solution